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Quantum Stochastic Differential Inclusions of Hypermaximal Monotone Type

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In continuation of our study of the existence of solutions of quantum stochastic differential inclusions, we first introduce and develop some aspects of the theory of maximal [resp. hypermaximal] monotone multifunctions, including the description of a number of properties of their resolvents and Yosida approximations, in the present noncommutative setting. Then, it is proved that, under a certain continuity assumption, a quantum stochastic differential inclusion of hypermaximal monotone type has a unique adapted solution which is obtained as the limit of the unique adapted solutions of a one-parameter family of Lipschitzian quantum stochastic differential equations. As examples, we show that a large class of quantum stochastic differential inclusions which satisfy the assumptions and conclusion of our main result arises as perturbations of certain quantum stochastic differential equations by some multivalued stochastic processes.

1. INTRODUCTION

In Ekhaguere (1992), we introduced the notion of a *quantum stochastic differential inclusion* within the framework of the Hudson and Parthasarathy (1984) formulation of quantum stochastic calculus. Inclusions are particularly relevant in, for example, *quantum stochastic control theory*, since control-theoretic problems may often be formulated as inclusions. In Ekhaguere (1992), the existence of solutions of a *Lipschitzian* quantum stochastic differential inclusion was established. Moreover, a *relaxation theorem* giving the relationship between the solutions of such an inclusion and those of its convexification was also proved. Relaxation theorems are important in control theory.

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This paper continues our study of the question of existence of solutions of quantum stochastic differential inclusions. We introduce the important class of inclusions of *hypermaximal monotone* type. The members of this class are interesting in the discussion of the *nonlinear evolution* of the systems described by them, since they generate *nonlinear evolution operators*: these will be investigated elsewhere. We establish that, subject to a certain continuity condition, every quantum stochastic differential inclusion of hypermaximal monotone type has a unique adapted solution.

The following is an outline of the rest of the paper. In Section 2, some of the notation employed in the subsequent discussion is clarified. This is made as consistent as possible with the notation in Ekhaguere (1992). Section 3 introduces the notion of a *regular* multifunction as well as various notions of monotonicity for such a map. Some aspects of the theory of *maximal* [resp. *hypermaximal*] *monotone* multifunctions are developed there, by proving two results concerning these classes of multifunctions. The results generalize their well-known counterparts in the Banach space context to the present noncommutative setting. We conclude the section by describing the class of hypermaximal monotone multifunctions that are employed in the subsequent discussion. In Section 4, we introduce the notions of the *resolvent* and *Yosida approximation* of a hypermaximal monotone multifunction lying in the class described in Section 3. These are single-valued maps. Theorem 4.1 gives a number of results concerning the maps. The quantum stochastic differential inclusion studied in this paper is introduced in Section 5 as Problem $(5.1)_0$. The results of Section 4 enable us to associate to Problem $(5.1)_0$ a one-parameter family of quantum stochastic differential *equations*. These are *Lipschitzian* equations, each of which possesses a unique adapted solution. Our main result is obtained by showing that the one-parameter family of solutions of the Lipschitzian quantum stochastic differential equations converges to a unique adapted solution of Problem $(5.1)_0$. As examples, we show that a large class of quantum stochastic differential inclusions which satisfy the assumptions and conclusion of our main result arises as perturbations of certain quantum stochastic differential equations by some multivalued stochastic processes. The results of this paper generalize classical analogs in the Banach space context and apply, in particular, to quantum stochastic differential *equations* of hypermaximal monotone type.

2. PRELIMINARIES

This section outlines some of the notation of this paper, which will be as consistent as possible with that in Ekhaguere (1992). Thus Υ is a fixed Hilbert space. To this space, we associate a number of other function spaces as follows. For $I \subseteq \mathbb{R}_+ \equiv [0, \infty)$, $L^2_\Upsilon(I)$ is the linear space of square-integrable

Y-valued functions on I, and $L^{\infty}_{Y,\text{loc}}(I)$ [resp. $L^{\infty}_{B(Y),\text{loc}}(I)$] is the linear space of all measurable, locally bounded functions from I to Y [resp. to B(Y), the Banach space of bounded linear maps on Y]. If $f \in L^{\infty}_{Y,\text{loc}}(I)$ and $\pi \in L^{\infty}_{B(Y),\text{loc}}(I)$, then $\pi f \in L^{\infty}_{Y,\text{loc}}(I)$ is given by $(\pi f)(t) = \pi(t)f(t)$, for almost every $t \in I$.

When **D** is some complex pre-Hilbert space with **H** as its completion, we write $L^{+}_{w}(\mathbf{D}, \mathbf{H})$ for the linear space of all linear maps x from **D** into **H** such that the domain of the operator adjoint x^* of x contains **D**, and $\Gamma(\mathbf{H})$ for the Fock space (Guichardet, 1972) over **H**. For $f \in \mathbf{H}$, define $\otimes^0 f = 1$ and if $n \geq 1$, define $\otimes^n f$ as the n-fold tensor product of f with itself. Then,

$$e(f) = \bigoplus_{n=0}^{\infty} (n!)^{-1/2} \otimes^n f$$

is in $\Gamma(\mathbf{H})$ and is the *exponential vector* associated with f. In $\Gamma(\mathbf{H})$, the set of all exponential vectors generates a dense subspace. Other properties of these vectors are described in Guichardet (1972) and Hudson and Parthasarathy (1984).

In the sequel, $\mathbb{D}$ is a pre-Hilbert space whose completion is $\mathfrak{R}$, and $\mathbb{E}$, $\mathbb{E}_t$, and $\mathbb{E}^t$, $t > 0$, are the linear spaces generated by the exponential vectors in $\Gamma(L^2_Y(\mathbb{R}_+))$, $\Gamma(L^2_Y([0, t)))$, and $\Gamma(L^2_Y([t, \infty)))$, $t > 0$, respectively. We denote the *inner product* and *norm* of the Hilbert space $\mathfrak{R} \otimes \Gamma(L^2_Y(\mathbb{R}_+))$ by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively, and write $\mathcal{A}$, $\mathcal{A}_t$, and $\mathcal{A}^t$ for the linear spaces of linear operators defined as follows:

$$\mathcal{A} \equiv L^{+}_{w}(\mathbb{D} \underline{\otimes} \mathbb{E}, \mathfrak{R} \otimes \Gamma(L^2_Y(\mathbb{R}_+)))$$

$$\mathcal{A}_t \equiv L^{+}_{w}(\mathbb{D} \underline{\otimes} \mathbb{E}_t, \mathfrak{R} \otimes \Gamma(L^2_Y([0, t)))) \otimes \mathbb{1}^t$$

$$\mathcal{A}^t \equiv \mathbb{1}_t \otimes L^{+}_{w}(\mathbb{E}^t, \Gamma(L^2_Y([t, \infty)))), \qquad t > 0$$

where $\underline{\otimes}$ denotes *algebraic tensor product* throughout the paper and $\mathbb{1}_t$ (resp. $\mathbb{1}^t$) is the identity map on $\mathfrak{R} \otimes \Gamma(L^2_Y([0, t)))$ [resp. $\Gamma(L^2_Y([t, \infty)))$], $t > 0$. It is clear that $\mathcal{A}_t$ and $\mathcal{A}^t$, $t > 0$, are linear subspaces of $\mathcal{A}$. The latter will be topologized as follows. For $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, define the seminorm $\|\cdot\|_{\eta,\xi}$ on $\mathcal{A}$ by

$$\|x\|_{\eta,\xi} = |\langle \eta, x\xi \rangle|, \qquad x \in \mathcal{A}, \quad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

and write τ_w for the locally convex topology generated by the family $\{\|\cdot\|_{\eta,\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$. The completions of the locally convex spaces $(\mathcal{A}, \tau_w)$, $(\mathcal{A}_t, \tau_w)$, and $(\mathcal{A}^t, \tau_w)$, $t > 0$, will be denoted by $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}_t$, and $\tilde{\mathcal{A}}^t$, $t > 0$, respectively. The net $\{\tilde{\mathcal{A}}_t: t \in \mathbb{R}_+\}$ filters $\tilde{\mathcal{A}}$.

In the sequel, we denote the Fock space $\Gamma(L^2_{\mathbb{Y}}(\mathbb{R}_+))$ simply by Γ and write $\mathbb{1}$ for the identity map on $\mathfrak{R} \otimes \Gamma$. Then, $\mathbb{1}$ is the unit of $\mathcal{A}$ (regarded as a partial *-algebra).

2.1. Stochastic Processes

Let $I \subseteq \mathbb{R}_+$ be a subinterval. A *stochastic process* indexed by I is an $\tilde{\mathcal{A}}$-valued map on I. Such a map X will be called *adapted* if $X(t) \in \tilde{\mathcal{A}}_t$ for each $t \in I$. We denote the set of all adapted processes indexed by $\mathbb{R}_+$ by $\mathrm{Ad}(\tilde{\mathcal{A}})$. We are interested in certain classes of members of $\mathrm{Ad}(\tilde{\mathcal{A}})$. We call $X \in \mathrm{Ad}(\tilde{\mathcal{A}})$ *weakly absolutely continuous* if the map $t \mapsto \langle \eta, X(t)\xi \rangle$, $t \in \mathbb{R}_+$, is absolutely continuous for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, and *locally absolutely square integrable* if $\|X(\cdot)\|^2_{\eta,\xi}$ is Lebesgue-measurable and integrable on $[0, t)$ for each $t \in \mathbb{R}_+$, all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$; the classes of stochastic processes determined by these notions will be denoted by $\mathrm{Ad}(\tilde{\mathcal{A}})_{\mathrm{wac}}$ and $L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})$, respectively. We return to the discussion of stochastic processes in later sections.

2.2. Other Notation

The following notation will also be employed.

If $\mathcal{X}$ is a linear space and n a natural number, we write $\mathcal{X}^n$ [resp. $\mathcal{X}^{(n)}$] for the n-fold *Cartesian* product [resp. n-fold *algebraic tensor product*] of $\mathcal{X}$ with itself. In case $\mathcal{X}$ is a Hilbert space, then $\mathcal{X}^{(n)}$ is the n-fold Hilbert space tensor product (Reed and Simon, 1972) of $\mathcal{X}$ with itself.

The set of all *sesquilinear forms* on $\mathcal{X}$ will be denoted by $\mathrm{sesq}(\mathcal{X})$. If $p \in \mathrm{sesq}(\mathcal{X})$, then the value of p at $(x, y) \in \mathcal{X}^2$ will be denoted by $p(x, y)$. Throughout, a sesquilinear form is assumed to be conjugate-linear on the left.

The sum $A + B$ of two subsets A and B of a linear space is defined by

$$A + B \equiv \{a + b: a \in A, b \in B\}$$

In particular, if a is some fixed member of the linear space, then

$$a + B \equiv \{a + b: b \in B\}$$

3. MONOTONE MULTIFUNCTIONS

Central to much of the subsequent discussion is the notion of a *multifunction*.

Let $\mathcal{X}$ and $\mathcal{Y}$ be sets. A map $P: \mathcal{X} \to 2^{\mathcal{Y}}$ is called a *multifunction* (or a *set-valued function* or a *multivalued function*). The subset $P(x) \subseteq \mathcal{Y}$ is the *image* or *value* of P at $x \in \mathcal{X}$. The values of the multifunctions encountered in this paper are *quadratic forms* (Reed and Simon, 1972).

The *domain* dom(P), *range* range(P), and *graph* graph(P) of P: $\mathscr{X} \to 2^{\mathscr{Y}}$ are defined as follows:

$$\operatorname{dom}(P) = \{x \in \mathscr{X}: P(x) \neq \emptyset\}$$
$$\operatorname{range}(P) = \cup_{x \in \mathscr{X}} P(x)$$
$$\operatorname{graph}(P) = \{(x, y) \in \mathscr{X} \times \mathscr{Y}: y \in P(x)\}$$

If dom(P) $= \mathscr{X}$, then P is called *strict*. For simplicity, we shall deal mainly with *strict* multifunctions.

A *selection* of a multifunction P: $\mathscr{X} \to 2^{\mathscr{Y}}$ is a map φ: $\mathscr{X} \to \mathscr{Y}$ such that

$$\varphi(x) \in P(x), \qquad \text{for each} \quad x \in \mathscr{X}$$

Depending on the structures on $\mathscr{X}$ and $\mathscr{Y}$, it is often of interest to find out whether a given multifunction has selections of some specified type: e.g., *measurable*, *continuous*, *Lipschitzian*, or *integrable* selections. For some orientation about this problem, see Michael (1956) and Parthasarathy (1972).

In case $\mathscr{Y}$ is a topological linear space, the values of the multifunction P: $\mathscr{X} \mapsto 2^{\mathscr{Y}}$ are called *convex* [resp. *closed*] if $P(x)$ is convex [resp. closed] for each $x \in \mathscr{X}$.

Let P and Q be two multifunctions from the set $\mathscr{X}$ to $2^{\mathscr{Y}}$. Then, the sum $P + Q$ is defined as follows:

$$(P + Q)(x) = \begin{cases} P(x) + Q(x) & \text{if neither } P(x) \text{ nor } Q(x) \text{ is empty} \\ \emptyset & \text{otherwise} \end{cases}$$

$x \in \mathscr{X}$.

The following notation will be repeatedly used. Let $\mathscr{X}$ be a set, $\mathscr{Y}$ a linear space, y_0 some member of $\mathscr{Y}$, and P: $\mathscr{X} \mapsto 2^{\mathscr{Y}}$. Then, we define $P(x) \otimes y_0$, $x \in \mathscr{X}$, by

$$P(x) \otimes y_0 = \{p \otimes y_0: p \in P(x)\}, \qquad x \in \mathscr{X}$$

and denote the multifunction $x \mapsto P(x) \otimes y_0$ from $\mathscr{X}$ into $2^{\mathscr{Y} \underline{\otimes} \mathscr{Y}}$ by $P \otimes y_0$.

3.1. Regular Multifunctions

We are principally interested in certain classes of multifunctions which are *monotone* in some sense. We introduce the relevant notions of monotonicity and develop some aspects of the theory of such monotone multifunctions.

In what follows, let $\tilde{\mathfrak{Z}} = \tilde{\mathscr{A}}$ or $\mathbb{R}_+ \times \tilde{\mathscr{A}}$.

The multifunctions considered in the sequel are maps from $\tilde{\mathfrak{Z}}$ into $2^{\operatorname{sesq}((\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)})}$. It follows that if $\mathscr{P}$ is such a multifunction and $z \in \tilde{\mathfrak{Z}}$, then $\mathscr{P}(z)$ is a set of *quadratic forms* on $(\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)}$. For $\zeta_1, \zeta_2 \in (\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)}$ and

$z \in \tilde{\mathfrak{Z}}$, we define $\mathcal{P}(z)(\zeta_1, \zeta_2)$ as the set

$$\mathcal{P}(z)(\zeta_1, \zeta_2) = \{p(\zeta_1, \zeta_2): p \text{ is a sesquilinear form on } (\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)} \text{ and } p \in \mathcal{P}(z)\}$$

and regard this as the value of $\mathcal{P}(z)$ at the point $(\zeta_1, \zeta_2) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)} \times (\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)}$.

We consider multifunctions from $\tilde{\mathfrak{Z}}$ into $2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$ that are *regular* in some sense. Such quadratic-forms-valued multifunctions are encountered in quantum stochastic calculus.

In the sequel, $\langle\cdot, \cdot\rangle_{(2)}$ denotes the inner product of $(\mathfrak{R} \otimes \Gamma)^{(2)}$.

If $\mathcal{B}$ is a subset of $\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}}$, then the notation $\langle\zeta_1, \mathcal{B}\zeta_2\rangle_{(2)}$ is specified by

$$\langle\zeta_1, \mathcal{B}\zeta_2\rangle_{(2)} = \{\langle\zeta_1, b\zeta_2\rangle_{(2)}: b \in \mathcal{B}\}$$

for $\zeta_1, \zeta_2 \in (\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)}$.

Definition 3.1. A multifunction $\mathcal{P}: \tilde{\mathfrak{Z}} \to 2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$ will be called *regular* if

$$\mathcal{P}(z)(\eta_1 \otimes \eta_2, \xi_1 \otimes \xi_2) = \langle\eta_1 \otimes \eta_2, \mathcal{P}_{\alpha_1\beta_1\alpha_2\beta_2}(z)(\xi_1 \otimes \xi_2)\rangle_{(2)}$$

for some subset $\mathcal{P}_{\alpha_1\beta_1\alpha_2\beta_2}(z)$ of $\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}}$, and all $z \in \tilde{\mathfrak{Z}}$, $\eta_j = u_j \otimes e(\alpha_j)$, $\xi = v_j \otimes e(\beta_j)$, $u_j, v_j \in \mathbb{D}$, $\alpha_j, \beta_j \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $j = 1, 2$.

Remark 3.2. 1. A regular multifunction $\mathcal{P}: \tilde{\mathcal{A}} \to 2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)}}$ gives rise to an array $(\mathcal{P}_{\alpha_1\beta_1\alpha_2\beta_2}: \alpha_j, \beta_j \in L^2_{\gamma}(\mathbb{R}_+), j = 1, 2)$ of multifunctions from $\tilde{\mathfrak{Z}}$ into $2^{\tilde{\mathcal{A}}\underline{\otimes}\tilde{\mathcal{A}}}$. The notions of monotonicity introduced below involve the diagonal $(\mathcal{P}_{\alpha\beta\alpha\beta}: \alpha, \beta \in L^2_{\gamma}(\mathbb{R}_+))$ of this array.

We assume in what follows that the range of $\mathcal{P}_{\alpha\beta\alpha\beta}$ *is contained in some unital subspace* $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$ *of* $\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}}$, *i.e., a subspace containing the unit* $1 \otimes 1$ *of* $\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}}$.

2. If $\mathcal{P}: \tilde{\mathfrak{Z}} \to 2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$ is *regular* and such that $\mathcal{P}_{\alpha\beta\alpha\beta}$ is of the form $P_{\alpha\beta} \otimes 1$, for some multifunction $P_{\alpha\beta}: \tilde{\mathfrak{Z}} \to 2^{\tilde{\mathcal{A}}}$, then we define the map

$$P: \tilde{\mathfrak{Z}} \to 2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$$

by

$$P(z)(\eta, \xi) = \langle\eta, P_{\alpha\beta}(z)\xi\rangle, \qquad z \in \tilde{\mathfrak{Z}}$$

for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, and write $\mathcal{P}$ as $\mathcal{P} = P \otimes 1$. In the sequel, *we assume that the range of* $P_{\alpha\beta}$ *is contained in some unital subspace* $\tilde{\mathcal{A}}_{\alpha\beta}$ *of* $\tilde{\mathcal{A}}$, *i.e., a subspace containing the unit* 1 *of* $\tilde{\mathcal{A}}$.

3.2. The Φ-Functional

For any ordered pair (η, ξ) in $(\mathbb{D} \underline{\otimes} \mathbb{E})^2$, the symbol $(\mathfrak{R} \otimes \Gamma)^{(2)}_{(\eta,\xi)}$ denotes the closure of the linear space $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})(\xi \otimes \eta)$ in $(\mathfrak{R} \otimes \Gamma)^{(2)}$ and $\Phi_{(\eta,\xi)}(\cdot, \cdot)$ is the map from $\tilde{\mathcal{A}} \times \tilde{\mathcal{A}}$ to $(\mathfrak{R} \otimes \Gamma)^{(2)}$ defined by

$$\Phi_{(\eta,\xi)}(x, y) = \eta \otimes (x - y)\xi, \qquad x, y \in \tilde{\mathcal{A}}$$

Then, $(\mathfrak{R} \otimes \Gamma)^{(2)}_{(\eta,\xi)}$ is a Hilbert subspace of $(\mathfrak{R} \otimes \Gamma)^{(2)}$ and the map $\Phi_{(\eta,\xi)}(\cdot, \cdot)$ is a *global Φ-system* for the pair $(\tilde{\mathcal{A}}, (\mathfrak{R} \otimes \Gamma)^{(2)}_{(\eta,\xi)})$ over $\tilde{\mathcal{A}}$, in the terminology of Browder (1976).

3.3. Notions of Monotonicity

The notions of monotonicity employed in this paper are introduced as follows.

Definition 3.3. A *regular* multifunction $\mathcal{P}: \tilde{\mathcal{A}} \to 2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$ will be called:

(i) *monotone* if for any ordered pair $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, the multifunction $x \mapsto \mathcal{P}_{\alpha\beta\alpha\beta}(x)(\xi \otimes \eta)$ from $\tilde{\mathcal{A}}$ to $2^{(\mathfrak{R}\otimes\Gamma)^{(2)}_{(\eta,\xi)}}$ is $\Phi_{(\eta,\xi)}$-monotone in the sense of Browder (1976), i.e., if

$$\mathrm{Re}(\langle (a - b)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)}) \geq 0$$

whenever $a \in \mathcal{P}_{\alpha\beta\alpha\beta}(x)$, $b \in \mathcal{P}_{\alpha\beta\alpha\beta}(y)$, and $x, y \in \tilde{\mathcal{A}}$, where $\mathrm{Re}(\cdots)$ denotes the *real part* of $(\cdots)$;

(ii) *maximal monotone* if $\mathcal{P}$ is monotone and for any ordered pair (η, ξ) in $(\mathbb{D} \underline{\otimes} \mathbb{E})^2$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, the multifunction $x \mapsto \mathcal{P}_{\alpha\beta\alpha\beta}(x)(\xi \otimes \eta)$ from $\tilde{\mathcal{A}}$ to $2^{(\mathfrak{R}\otimes\Gamma)^{(2)}_{(\eta,\xi)}}$ is maximal $\Phi_{(\eta,\xi)}$-monotone in the sense of Browder (1976), i.e., if the graph of the multifunction $x \mapsto \mathcal{P}_{\alpha\beta\alpha\beta}(x)(\xi \otimes \eta)$ is not properly contained in the graph of any other monotone multifunction;

(iii) *hypermaximal monotone* if $\mathcal{P}$ is monotone and for arbitrary $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, there is a single-valued monotone map $K_{\alpha\beta}$ from $\tilde{\mathcal{A}}$ to $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$ such that the multifunction $x \mapsto \mathcal{P}_{\alpha\beta\alpha\beta}(x)$ from $\tilde{\mathcal{A}}$ into $2^{(\tilde{\mathcal{A}}\underline{\otimes}\tilde{\mathcal{A}})_{\alpha\beta}}$ satisfies the following two conditions: (a) the range of $K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta}$ is all of $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$; and (b) $(K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta})^{-1}$ is a continuous single-valued map from $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$ to $\tilde{\mathcal{A}}$.

Remark. 1. The multifunctions encountered in the subsequent discussion are in general regular maps $\mathcal{P}: \mathbb{R}_+ \times \tilde{\mathcal{A}} \to 2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$. These will be called *monotone* [resp. *maximal monotone*; resp. *hypermaximal monotone*] if the multifunction $x \mapsto \mathcal{P}(t, x)$, $x \in \tilde{\mathcal{A}}$, is monotone [resp. maximal monotone; resp. hypermaximal monotone] for each $t \in \mathbb{R}_+$.

2. In addition to the weak topology τ_w already introduced, we shall also consider the strong topology τ_s on $\mathcal{A}$. This is the locally convex topology whose family $\{\|\cdot\|_\xi: \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ of seminorms is specified by

$$\|x\|_\xi = \|x\xi\|, \qquad x \in \mathcal{A}, \quad \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

3. In what follows, we prove two results involving the above notions of monotonicity. Analogs of these results are well known in the context of Banach spaces (Browder, 1976).

Proposition 3.4. Let $\mathcal{P}$ be a regular multifunction from $\tilde{\mathcal{A}}$ into $2^{\text{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$ and $\alpha, \beta \in L^\infty_{\Upsilon,\text{loc}}(\mathbb{R}_+)$. Then:

(i) The multifunction $\mathcal{P}_{\alpha\beta\alpha\beta}$ has convex and τ_w-closed values in $2^{(\tilde{\mathcal{A}}\underline{\otimes}\tilde{\mathcal{A}})_{\alpha\beta}}$.

(ii)(a) If $x \in \tilde{\mathcal{A}}$, $a \in (\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$, $\{x_\delta: \delta \in \Delta\}$ is a net that τ_s-converges to x, $a_\delta \in \mathcal{P}_{\alpha\beta\alpha\beta}(x_\delta)$, and the net $\{a_\delta: \delta \in \Delta\}$ τ_w-converges to a, then $a \in \mathcal{P}_{\alpha\beta\alpha\beta}(x)$.

(ii)(b) If $\mathcal{P}$ is of the form $\mathcal{P} = P \otimes \mathbb{1}$, $x \in \tilde{\mathcal{A}}$, $a \in \tilde{\mathcal{A}}_{\alpha\beta} \otimes \mathbb{1}$, $\{x_\delta: \delta \in \Delta\}$ is a net that τ_w-converges to x, $a_\delta \in \mathcal{P}_{\alpha\beta\alpha\beta}(x_\delta)$, and the net $\{a_\delta: \delta \in \Delta\}$ τ_w-converges to a, then $a \in \mathcal{P}_{\alpha\beta\alpha\beta}(x)$.

Proof. (i) By the definition of a maximal monotone multifunction, a lies in $\mathcal{P}_{\alpha\beta\alpha\beta}(x)$ iff

$$\text{Re}(\langle (a - b)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)}) \geq 0$$

for all (η, ξ) in $(\mathbb{D} \underline{\otimes} \mathbb{E})^2$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $u, v \in \mathbb{D}$, and all $(y, b) \in \tilde{\mathcal{A}} \times (\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$ with $b \in \mathcal{P}_{\alpha\beta\alpha\beta}(y)$. As the set

$$\mathcal{P}_{\alpha\beta\alpha\beta,yb}(x) = \{a \in (\tilde{\mathcal{A}} \underline{\otimes} \mathcal{A})_{\alpha\beta}: \text{Re}(\langle (a - b)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)}) \geq 0$$
$$\forall (\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2, \text{ with}$$
$$\eta = u \otimes e(\alpha), \xi = v \otimes e(\beta), u, v \in \mathbb{D}\}$$

is convex and τ_w-closed, and

$$\mathcal{P}_{\alpha\beta\alpha\beta}(x) = \bigcap\{\mathcal{P}_{\alpha\beta\alpha\beta,yb}(x): (y, b) \in \tilde{\mathcal{A}} \times (\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta} \text{ with } b \in \mathcal{P}_{\alpha\beta\alpha\beta}(y)\}$$

it follows that $\mathcal{P}_{\alpha\beta\alpha\beta}(x)$ is also convex and τ_w-closed.

(ii)(a) By the monotonicity of $\mathcal{P}$, if $(y, b) \in \tilde{\mathcal{A}} \times (\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$, with $b \in \mathcal{P}_{\alpha\beta\alpha\beta}(y)$, then

$$\text{Re}(\langle (a_\delta - b)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_\delta, y)\rangle_{(2)}) \geq 0 \qquad (\star)$$

for each $\delta \in \Delta$ and all pairs $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $u, v \in \mathbb{D}$. By the definition of $\Phi_{(\eta,\xi)}(\cdot, \cdot)$, the net $\{\Phi_{(\eta,\xi)}(x_\delta, y)\colon \delta \in \Delta\}$ converges in $(\mathfrak{R} \otimes \Gamma)^{(2)}$ to $\Phi_{(\eta,\xi)}(x, y)$ whenever the net $\{x_\delta\colon \delta \in \Delta\}$ τ_s-converges to $x \in \tilde{\mathscr{A}}$. Furthermore, the net $\{a_\delta - b\colon \delta \in \Delta\}$ τ_w-converges to $a - b$ in $\tilde{\mathscr{A}} \underline{\otimes} \tilde{\mathscr{A}}$ whenever the net $\{a_\delta\colon \delta \in \Delta\}$ τ_w-converges to a in $\tilde{\mathscr{A}} \underline{\otimes} \tilde{\mathscr{A}}$. Hence, taking limits in $(\star)$ gives

$$\mathrm{Re}(\langle (a - b)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)}) \geq 0$$

for all $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $u, v \in \mathbb{D}$. By the maximal monotone nature of $\mathcal{P}$, it follows that $a \in \mathcal{P}_{\alpha\beta\alpha\beta}(x)$.

(ii)(b) The proof is essentially as in (ii)(a), noting that as $\mathcal{P} = P \otimes \mathbb{1}$, the set $\mathcal{P}_{\alpha\beta\alpha\beta}(x)$, $x \in \tilde{\mathscr{A}}$, has the form $\mathcal{P}_{\alpha\beta\alpha\beta}(x) = P_{\alpha\beta}(x) \otimes \mathbb{1}$, $x \in \tilde{\mathscr{A}}$. Hence, if $a_\delta \in P_{\alpha\beta}(x_\delta) \otimes \mathbb{1}$ and $(y, b) \in \tilde{\mathscr{A}} \times (\tilde{\mathscr{A}} \otimes \mathbb{1})$, with $b \in P_{\alpha\beta}(y) \otimes \mathbb{1}$, then there are $\tilde{a}_\delta \in P_{\alpha\beta}(x_\delta)$ and $\tilde{b} \in P_{\alpha\beta}(y)$ such that $a = \tilde{a}_\delta \otimes \mathbb{1}$ and $b = \tilde{b}_\delta \otimes \mathbb{1}$. Hence $(\star)$ above reduces to

$$\begin{aligned}&\mathrm{Re}(\langle ((\tilde{a}_\delta - \tilde{b}) \otimes \mathbb{1})(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_\delta, y)\rangle_{(2)})\\ &\quad = \mathrm{Re}(\langle (\tilde{a}_\delta - \tilde{b})\xi, \eta\rangle\langle \eta, (x_\delta - y)\xi\rangle)\end{aligned}$$

The assertion is deduced from this.

This concludes the proof. ■

Theorem 3.5. A regular, hypermaximal monotone multifunction

$$\mathcal{P}\colon \tilde{\mathscr{A}} \mapsto 2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$$

is maximal monotone.

Proof. The following proof extends the main arguments of Theorem 3.9 in Browder (1976) to the present setting.

Let $\xi \otimes \eta \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. The claim will be established by showing that whenever $(y, b) \in \tilde{\mathscr{A}} \times (\tilde{\mathscr{A}} \underline{\otimes} \tilde{\mathscr{A}})_{\alpha\beta}$ and

$$\mathrm{Re}(\langle (a - b)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)}) \geq 0$$

for all $(x, a) \in \tilde{\mathscr{A}} \times (\tilde{\mathscr{A}} \underline{\otimes} \tilde{\mathscr{A}})_{\alpha\beta}$, with $a \in \mathcal{P}_{\alpha\beta\alpha\beta}(x)$, then $b \in \mathcal{P}_{\alpha\beta\alpha\beta}(y)$.

As $\mathcal{P}$ is hypermaximal monotone, there is a monotone single-valued map $K_{\alpha\beta}\colon \tilde{\mathscr{A}} \to (\tilde{\mathscr{A}} \underline{\otimes} \tilde{\mathscr{A}})_{\alpha\beta}$ such that the range of $K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta}$ is all of $(\tilde{\mathscr{A}} \underline{\otimes} \tilde{\mathscr{A}})_{\alpha\beta}$ and $(K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta})^{-1}$ is a continuous single-valued map from $(\tilde{\mathscr{A}} \underline{\otimes} \tilde{\mathscr{A}})_{\alpha\beta}$ to $\tilde{\mathscr{A}}$. By the monotonicity of $K_{\alpha\beta}$,

$$\mathrm{Re}(\langle (K_{\alpha\beta}(x) - K_{\alpha\beta}(y))(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)}) \geq 0$$

Adding the last two inequalities gives

$$\mathrm{Re}(\langle(a + K_{\alpha\beta}(x) - b - K_{\alpha\beta}(y))(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)}) \geq 0 \quad (3.1)$$

Let c be a fixed but otherwise arbitrary member of $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$ and $t \in (0, 1)$. Define d_t and x_t by

$$d_t = b + K_{\alpha\beta}(y) + tc$$

$$x_t = (K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta})^{-1}(d_t)$$

Then, $d_t \in K_{\alpha\beta}(x_t) + \mathcal{P}_{\alpha\beta\alpha\beta}(x_t)$, whence

$$d_t = K_{\alpha\beta}(x_t) + q_t \qquad \text{for some} \quad q_t \in \mathcal{P}_{\alpha\beta\alpha\beta}(x_t) \quad (3.2)$$

Replacing the pair (x, a) in (3.1) by (x_t, q_t) and using (3.2), one gets

$$\mathrm{Re}(\langle c(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_t, y)\rangle_{(2)}) \geq 0, \qquad \forall(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2 \quad (3.3)$$

When $t \downarrow 0$, d_t τ_w-converges to $b + K_{\alpha\beta}(y)$ in $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$. By the hypermaximal monotonicity of $\mathcal{P}$, the map $(K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta})^{-1}$ is continuous from $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$ to $\tilde{\mathcal{A}}$. Hence, $x_t = (K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta})^{-1}(d_t)$ converges to $(K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta})^{-1}(b + K_{\alpha\beta}(y)) \equiv x_0$ in $\tilde{\mathcal{A}}$ as $t \downarrow 0$. Since c can be represented as a finite sum $c = \sum_j c_{1j} \otimes c_{2j}$, one sees that

$$\begin{aligned}&\left|\langle c(\xi \otimes \eta, \Phi_{(\eta,\xi)}(x_t, y)\rangle_{(2)} - \langle c(\xi \otimes \eta, \Phi_{(\eta,\xi)}(x_0, y)\rangle_{(2)}\right| \\ &\quad \leq \sum_j \left|\langle c_{1j}\xi, \eta\rangle\right| \left|\langle c_{2j}\eta, (x_t - x_0)\xi\rangle\right|\end{aligned}$$

Hence, allowing $t \downarrow 0$ in (3.3) gives

$$\mathrm{Re}(\langle c(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_0, y)\rangle_{(2)}) \geq 0, \qquad \forall(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2 \quad (3.4)$$

Since c is arbitrary, replacing c by $-c$ does not change the inequality in (3.4). Hence

$$\mathrm{Re}(\langle c(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_0, y)\rangle_{(2)}) = 0, \qquad \forall(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2 \quad (3.5)$$

Analogously, replacing c in (3.4) by ic or $-ic$ (with $i = \sqrt{-1}$) does not change the inequality in (3.4). Since $\langle \cdot, \cdot \rangle_{(2)}$ is conjugate-linear on the left, it follows that

$$\mathrm{Im}(\langle c(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_0, y)\rangle_{(2)}) = 0, \qquad \forall(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2 \quad (3.6)$$

where $\mathrm{Im}(\cdots)$ denotes the *imaginary part* of $(\cdots)$. Hence, from (3.5) and (3.6),

$$\langle c(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_0, y)\rangle_{(2)} = 0$$

for all $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$ and arbitrary c in $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$, whence $\Phi_{(\eta,\xi)}(x_0, y) = 0$ for all $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$, since $(\tilde{\mathcal{A}} \underline{\otimes} \tilde{\mathcal{A}})_{\alpha\beta}$ contains $1 \otimes 1$. This is equivalent to $\eta \otimes (x_0 - y)\xi = 0$ for all $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$, whence $x_0 = y$. Hence, $y = (K_{\alpha\beta} + \mathcal{P}_{\alpha\beta\alpha\beta})^{-1}(b + K_{\alpha\beta}(y))$, whence $b + K_{\alpha\beta}(y) \in K_{\alpha\beta}(y) + \mathcal{P}_{\alpha\beta\alpha\beta}(y)$, thereby forcing $b \in \mathcal{P}_{\alpha\beta\alpha\beta}(y)$. This concludes the proof. ■

3.4. The Class Hypmax($\mathbb{R}_+ \times \tilde{\mathcal{A}}$)

Let $\mathrm{id}_{\tilde{\mathcal{A}}}$ be the identity map on $\tilde{\mathcal{A}}$. Then, the single-valued map

$$\mathrm{id}_{\tilde{\mathcal{A}}}(\cdot) \otimes \mathbb{1}: \quad \tilde{\mathcal{A}} \to \tilde{\mathcal{A}} \otimes \mathbb{1}$$

defined by

$$x \mapsto \mathrm{id}_{\tilde{\mathcal{A}}}(x) \otimes \mathbb{1} = x \otimes \mathbb{1}, \qquad x \in \tilde{\mathcal{A}}$$

is *monotone* since

$$\langle (\mathrm{id}_{\tilde{\mathcal{A}}}(x) \otimes \mathbb{1} - \mathrm{id}_{\tilde{\mathcal{A}}}(y) \otimes \mathbb{1})(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, y)\rangle_{(2)} = \|x - y\|^2_{\eta,\xi}$$

for all $x, y \in \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

In our discussion of quantum stochastic differential inclusions in Section 5, the relevant class of multifunctions is defined as follows.

Definition 3.6. The class Hypmax($\mathbb{R}_+ \times \tilde{\mathcal{A}}$) is the set of all *regular* multifunctions $\mathcal{P}$: $\mathbb{R}_+ \times \tilde{\mathcal{A}} \to 2^{\mathrm{sesq}((\mathbb{D}\underline{\otimes}\mathbb{E})^{(2)})}$ with the following properties:

(i) Relative to the representation of $\mathcal{P}$ in Definition 3.1, the multifunction $\mathcal{P}_{\alpha\beta\alpha\beta}$ is, for arbitrary $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, given by

$$\mathcal{P}_{\alpha\beta\alpha\beta}(t, x) = P_{\alpha\beta}(t, x) \otimes \mathbb{1}, \qquad (t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}} \tag{3.7}$$

for some multifunction $P_{\alpha\beta}$: $\mathbb{R}_+ \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$, with range contained in a unital subspace $\tilde{\mathcal{A}}_{\alpha\beta}$ of $\tilde{\mathcal{A}}$.

(ii) $\mathcal{P}$ is *hypermaximal monotone*, with its associated *monotone*, continuous, single-valued map $K_{\alpha\beta}$ in Definition 3.3(iii) given by

$$K_{\alpha\beta}(x) = \mathrm{id}_{\tilde{\mathcal{A}}}(x) \otimes \mathbb{1}, \qquad x \in \tilde{\mathcal{A}} \tag{3.8}$$

for all $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$.

4. RESOLVENT AND YOSIDA APPROXIMATION

Let $\mathcal{P} \in$ Hypmax($\mathbb{R}_+ \times \tilde{\mathcal{A}}$). Then, by equation (3.7), for $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$,

$$\mathcal{P}_{\alpha\beta\alpha\beta}(t, x) = P_{\alpha\beta}(t, x) \otimes \mathbb{1}, \qquad (t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$$

As $\mathcal{P}$ is *hypermaximal monotone*, the multifunction

$$x \mapsto \mathrm{id}_{\tilde{\mathcal{A}}}(x) \otimes 1 + \lambda(P_{\alpha\beta}(t, x) \otimes 1), \qquad x \in \tilde{\mathcal{A}}$$

is a surjection onto $\tilde{\mathcal{A}}_{\alpha\beta} \otimes 1$ for each $t \in \mathbb{R}_+$, $\lambda > 0$; moreover, the map

$$(\mathrm{id}_{\tilde{\mathcal{A}}}(\cdot) \otimes 1 + \lambda(P_{\alpha\beta}(t, \cdot) \otimes 1))^{-1}$$

is both single-valued and continuous from $\tilde{\mathcal{A}}_{\alpha\beta} \otimes 1$ to $\tilde{\mathcal{A}}$, for each $t \in \mathbb{R}_+$, all $\lambda > 0$, and $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$. As

$$\mathrm{id}_{\tilde{\mathcal{A}}}(x) \otimes 1 + \lambda(P_{\alpha\beta}(t, x) \otimes 1) = (\mathrm{id}_{\tilde{\mathcal{A}}}(x) + \lambda P_{\alpha\beta}(t, x)) \otimes 1$$

$(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, it follows that the multifunction $x \mapsto \mathrm{id}_{\tilde{\mathcal{A}}}(x) + \lambda P_{\alpha\beta}(t, x)$, $x \in \tilde{\mathcal{A}}$, is a surjection onto $\tilde{\mathcal{A}}_{\alpha\beta}$ for each $t \in \mathbb{R}_+$; moreover, the map $(\mathrm{id}_{\tilde{\mathcal{A}}}(\cdot) + \lambda P_{\alpha\beta}(t, \cdot))^{-1}$ is both single-valued and continuous from $\tilde{\mathcal{A}}_{\alpha\beta}$ to $\tilde{\mathcal{A}}$, for each t $\in \mathbb{R}_+$, all $\lambda > 0$, and all $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$.

We introduce the following maps:

$$J_{\lambda,\alpha\beta}(t, \cdot) = (\mathrm{id}_{\tilde{\mathcal{A}}}(\cdot) + \lambda P_{\alpha\beta}(t, \cdot))^{-1}$$

$$P_{\lambda,\alpha\beta}(t, \cdot) = \frac{1}{\lambda}(\mathrm{id}_{\tilde{\mathcal{A}}}(\cdot) - J_{\lambda,\alpha\beta}(t, \cdot))$$

$t \in \mathbb{R}_+$, $\lambda > 0$, and $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$. These are single-valued maps. They give rise to the *quadratic forms* $J_\lambda(t, x)$ and $P_\lambda(t, x)$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, defined as follows:

$$J_\lambda(t, x)(\eta, \xi) = \langle \eta, J_{\lambda,\alpha\beta}(t, x)\xi \rangle$$

$$P_\lambda(t, x)(\eta, \xi) = \langle \eta, P_{\lambda,\alpha\beta}(t, x)\xi \rangle$$

$\lambda > 0$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

In terms of the maps just introduced, we define $\mathcal{J}_\lambda$ and $\mathcal{P}_\lambda$ by

$$\mathcal{J}_\lambda(t, x) = J_\lambda(t, x) \otimes 1$$

$$\mathcal{P}_\lambda(t, x) = P_\lambda(t, x) \otimes 1$$

$\lambda > 0$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$.

The single-valued map J_λ [resp. $\mathcal{J}_\lambda$] will be called the *resolvent* of the multifunction P [resp. $\mathcal{P}$] and the single-valued map P_λ [resp. $\mathcal{P}_\lambda$] will be called the *Yosida approximation* of the multifunction P [resp. $\mathcal{P}$].

Remark. 1. A number of the properties of the maps J_λ and P_λ that are employed below in the proof of Theorem 5.6 are described in this section. To this end, we shall use the following facts.

2. If $\mathcal{K}$ is a closed convex subset of $\mathfrak{R} \otimes \Gamma$, then the projector of best approximation $p_{\mathcal{K}}$ of $\mathfrak{R} \otimes \Gamma$ onto $\mathcal{K}$ is characterized by

$$\|\chi - p_{\mathcal{K}}(\chi)\| = \inf_{\theta \in \mathcal{K}} \|\chi - \theta\|, \qquad \chi \in \mathfrak{R} \otimes \Gamma$$

It follows that $p_{\mathcal{K}}(0)$ is the member of $\mathcal{K}$ with the least norm. Define $m(\mathcal{K})$ by

$$m(\mathcal{K}) = p_{\mathcal{K}}(0)$$

When $\mathcal{K} = \{\chi_0\}$, a singleton, then clearly $m(\mathcal{K}) = \chi_0$.

3. For $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, α, $\beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, and $\mathcal{P} \in \mathrm{Hypmax}(\mathbb{R}_+ \times \tilde{\mathcal{A}})$, it follows from Proposition 3.4(i) that $P_{\alpha\beta}(t, x)\xi$ is a closed convex subset of $\mathfrak{R} \otimes \Gamma$, and $m(P_{\alpha\beta}(t, x)\xi)$ lies in $P_{\alpha\beta}(t, x)\xi$. This gives rise to a map $m_{\alpha\beta}(t, x)$ from the set $\mathbb{D} \underline{\otimes} \mathbb{E}$ to $\mathfrak{R} \otimes \Gamma$ defined by

$$m_{\alpha\beta}(t, x)\xi = m(P_{\alpha\beta}(t, x)\xi), \qquad \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

As $m(P_{\alpha\beta}(t, x)\xi)$ lies in $P_{\alpha\beta}(t, x)\xi$ and every member of $P_{\alpha\beta}(t, x)\xi$ is of the form $z\xi$, for some $z \in P_{\alpha\beta}(t, x)$, the map $m_{\alpha\beta}(t, x)$ may be identified with a member of $P_{\alpha\beta}(t, x)$.

Theorem 4.1. Let $\mathcal{P} \in \mathrm{Hypmax}(\mathbb{R}_+ \times \tilde{\mathcal{A}})$, $\lambda > 0$, and α, $\beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$. Then:

1(i) For each $t \in \mathbb{R}_+$, the map $x \mapsto J_{\lambda,\alpha\beta}(t, x)$, $x \in \tilde{\mathcal{A}}$, is Lipschitzian with Lipschitz constant 1; (ii) for each $t \in \mathbb{R}_+$, the map $x \mapsto P_{\lambda,\alpha\beta}(t, x)$, $x \in \tilde{\mathcal{A}}$, is Lipschitzian with Lipschitz constant $1/\lambda$; (iii) $P_\lambda(t, x)(\eta, \xi) \in P(t, J_{\lambda,\alpha\beta}(t, x))(\eta, \xi)$, for all $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, α, $\beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, u, $v \in \mathbb{D}$; and (iv) for each $\lambda > 0$, the map $\mathcal{P}_\lambda = P_\lambda \otimes 1$ is monotone.

2. For arbitrary η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$,

$$\|P_{\lambda,\alpha\beta}(t, x) - m_{\alpha\beta}(t, x)\|^2_{\eta,\xi} \leq \|m_{\alpha\beta}(t, x)\|^2_{\eta,\xi} - \|P_{\lambda,\alpha\beta}(t, x)\|^2_{\eta,\xi}$$

3. As $\lambda \downarrow 0$, $J_\lambda(t, x)(\eta, \xi)$ converges to $\langle \eta, x\xi \rangle$ for arbitrary η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$.

Proof. 1(i) + (ii): Let $t \in \mathbb{R}_+$, $\lambda > 0$, α, $\beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$. As $x \mapsto J_{\lambda,\alpha\beta}(t, x)$, $x \in \tilde{\mathcal{A}}$, is surjective for each $t \in \mathbb{R}_+$, then given $y_j \in \tilde{\mathcal{A}}$, $j = 1, 2$, the inclusions

$$y_j \in x_j + \lambda P_{\alpha\beta}(t, x_j), \qquad j = 1, 2$$

can be solved for $x_j \in \tilde{\mathcal{A}}$, $j = 1, 2$. Hence, there exist $v_j \in P_{\alpha\beta}(t, x_j)$, $j = 1, 2$, such that

$$y_j = x_j + \lambda v_j, \qquad j = 1, 2$$

whence

$$\|y_1 - y_2\|^2_{\eta,\xi} = \|x_1 - x_2\|^2_{\eta,\xi} + \lambda^2\|v_1 - v_2\|^2_{\eta,\xi}$$
$$+ 2\lambda \operatorname{Re}(\langle((v_1 - v_2) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_1, x_2)\rangle_{(2)})$$
$$\geq \|x_1 - x_2\|^2_{\eta,\xi} + \lambda^2\|v_1 - v_2\|^2_{\eta,\xi}$$

for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{Y,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, since $\mathcal{P}$ is monotone. Hence

$$\|x_1 - x_2\|_{\eta,\xi} \leq \|y_1 - y_2\|_{\eta,\xi}, \qquad \forall \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

showing that for each $t \in \mathbb{R}_+$, the map $x \mapsto J_{\lambda,\alpha\beta}(t, x)$, $x \in \tilde{\mathcal{A}}$, is Lipschitzian with Lipschitz constant 1, and

$$\|v_1 - v_2\|_{\eta,\xi} \leq \frac{1}{\lambda} \|y_1 - y_2\|_{\eta,\xi}, \qquad \forall \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

showing that for each $t \in \mathbb{R}_+$, the map $x \mapsto P_{\lambda,\alpha\beta}(t, x)$, $x \in \tilde{\mathcal{A}}$, is Lipschitzian with Lipschitz constant $1/\lambda$.

(iii) By the definitions of $J_{\lambda,\alpha\beta}$ and $P_{\lambda,\alpha\beta}$, one gets

$$P_{\lambda,\alpha\beta}(t, x) = \frac{1}{\lambda} [x - J_{\lambda,\alpha\beta}(t, x)]$$
$$\in \frac{1}{\lambda} [J_{\lambda,\alpha\beta}(t, x) + \lambda P_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t, x))] - \frac{1}{\lambda} J_{\lambda,\alpha\beta}(t, x)$$
$$= P_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t, x))$$

whence

$$P_\lambda(t, x)(\eta, \xi) \in P(t, J_{\lambda,\alpha\beta}(t, x))(\eta, \xi)$$

for all $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{Y,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

(iv) To show that for each $\lambda > 0$, the single-valued map

$$\mathcal{P}_\lambda = P_\lambda \otimes 1\!: \quad \mathbb{R}_+ \times \tilde{\mathcal{A}} \to \operatorname{sesq}((\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)})$$

is *monotone*, let $x_1, x_2 \in \tilde{\mathcal{A}}$, $\alpha, \beta \in L^\infty_{Y,\mathrm{loc}}(\mathbb{R}_+)$, and $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$. Using

$$x_j = J_{\lambda,\alpha\beta}(t, x_j) + \lambda P_{\lambda,\alpha\beta}(t, x_j), \qquad t \in \mathbb{R}_+, \qquad j = 1, 2$$

we get

$$\operatorname{Re}(\langle((P_{\lambda,\alpha\beta}(t, x_1) - P_{\lambda,\alpha\beta}(t, x_2)) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x_1, x_2)\rangle_{(2)})$$
$$= \operatorname{Re}(\langle((P_{\lambda,\alpha\beta}(t, x_1) - P_{\lambda,\alpha\beta}(t, x_2)) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(J_{\lambda,\alpha\beta}(t, x_1),$$

$$
\begin{aligned}
&J_{\lambda,\alpha\beta}(t, x_2))\rangle_{(2)}) \\
&\quad + \lambda \operatorname{Re}(\langle((P_{\lambda,\alpha\beta}(t, x_1) - P_{\lambda,\alpha\beta}(t, x_2)) \otimes \mathbb{1})(\xi \otimes \eta), \Phi_{(\eta,\xi)}(P_{\lambda,\alpha\beta}(t, x_1), \\
&\quad P_{\lambda,\alpha\beta}(t, x_2))\rangle_{(2)}) \\
&= \operatorname{Re}(\langle((P_{\lambda,\alpha\beta}(t, x_1) - P_{\lambda,\alpha\beta}(t, x_2)) \otimes \mathbb{1})(\xi \otimes \eta), \Phi_{(\eta,\xi)}(J_{\lambda,\alpha\beta}(t, x_1), \\
&\quad J_{\lambda,\alpha\beta}(t, x_2))\rangle_{(2)}) \\
&\quad + \lambda \|P_{\lambda,\alpha\beta}(t, x_1) - P_{\lambda,\alpha\beta}(t, x_2)\|^2_{\eta,\xi} \\
&\geq 0, \qquad \text{since} \quad P_{\lambda,\alpha\beta}(t, x_j) \in P_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t, x_j)), \quad j = 1, 2
\end{aligned}
$$

showing that the single-valued map $\mathcal{P}_\lambda = P_\lambda \otimes \mathbb{1}: \mathbb{R}_+ \times \tilde{\mathcal{A}} \to \operatorname{sesq}((\mathbb{D} \underline{\otimes} \mathbb{E})^{(2)})$ is *monotone*, as claimed.

2. Let $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$ and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Then

$$
\begin{aligned}
&\|P_{\lambda,\alpha\beta}(t, x) - m_{\alpha\beta}(t, x)\|^2_{\eta,\xi} \\
&\quad = \|m_{\alpha\beta}(t, x)\|^2_{\eta,\xi} - \|P_{\lambda,\alpha\beta}(t, x)\|^2_{\eta,\xi} \\
&\qquad - 2 \operatorname{Re}(\langle(m_{\alpha\beta}(t, x) - P_{\lambda,\alpha\beta}(t, x))\xi, \eta\rangle\langle\eta, P_{\lambda,\alpha\beta}(t, x)\xi\rangle) \\
&\quad = \|m_{\alpha\beta}(t, x)\|^2_{\eta,\xi} - \|P_{\lambda,\alpha\beta}(t, x)\|^2_{\eta,\xi} \\
&\qquad - 2 \operatorname{Re}(\langle((m_{\alpha\beta}(t, x) - P_{\lambda,\alpha\beta}(t, x)) \otimes \mathbb{1})(\xi \otimes \eta), \eta \otimes P_{\lambda,\alpha\beta}(t, x)\xi\rangle_{(2)})
\end{aligned}
$$

Now, as $m_{\alpha\beta}(t, x) \in P_{\alpha\beta}(t, x)$, $P_{\lambda,\alpha\beta}(t, x) \in P_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t, x))$, $P_{\lambda,\alpha\beta}(t, x) = (1/\lambda)(x - J_{\lambda,\alpha\beta}(t, x))$, and $\mathcal{P}$ is *monotone*, it follows that

$$
\begin{aligned}
&\operatorname{Re}(\langle((m_{\alpha\beta}(t, x) - P_{\lambda,\alpha\beta}(t, x)) \otimes \mathbb{1})(\xi \otimes \eta), \eta \otimes P_{\lambda,\alpha\beta}(t, x)\xi\rangle_{(2)}) \\
&\quad = \frac{1}{\lambda} \operatorname{Re}(\langle((m_{\alpha\beta}(t, x) - P_{\lambda,\alpha\beta}(t, x)) \otimes \mathbb{1})(\xi \otimes \eta), \eta \otimes (x - J_{\lambda,\alpha\beta}(t, x))\xi\rangle_{(2)}) \\
&\quad = \frac{1}{\lambda} \operatorname{Re}(\langle((m_{\alpha\beta}(t, x) - P_{\lambda,\alpha\beta}(t, x)) \otimes \mathbb{1})(\xi \otimes \eta), \Phi_{(\eta,\xi)}(x, J_{\lambda,\alpha\beta}(t, x))\rangle_{(2)}) \\
&\quad \geq 0
\end{aligned}
$$

Hence

$$
\|P_{\lambda,\alpha\beta}(t, x) - m_{\alpha\beta}(t, x)\|^2_{\eta,\xi} \leq \|m_{\alpha\beta}(t, x)\|^2_{\eta,\xi} - \|P_{\lambda,\alpha\beta}(t, x)\|^2_{\eta,\xi}
$$

with the corollary that

$$
\|P_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi} \leq \|m_{\alpha\beta}(t, x)\|_{\eta,\xi}
$$

for all $\lambda > 0$, $\alpha, \beta \in L^\infty_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$.

3. As

$$\|x - J_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi} = \lambda\|P_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi} \leq \lambda\|m_{\alpha\beta}(t, x)\|_{\eta,\xi}$$

it follows that $J_\lambda(t, x)(\eta, \xi)$ converges to $\langle\eta, x\xi\rangle$ as $\lambda \downarrow 0$ for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$.

This concludes the proof. ∎

5. HYPERMAXIMAL MONOTONE INCLUSIONS

Let $a(f)$, $a^+(g)$, and $\lambda(\pi)$ be the *annihilation*, *creation*, and *gauge* operators of quantum field theory associated with $f, g \in L^2_\Upsilon(\mathbb{R}_+)$ and $\pi \in L^\infty_{B(\Upsilon),\mathrm{loc}}(\mathbb{R}_+)$ (Hudson and Parthasarathy, 1984). [Throughout, the triple (f, g, π) is assumed fixed.] Then define $A_f(t)$, $A_g^+(t)$, and $\Lambda_\pi(t)$ by

$$A_f(t) = a(f\chi_{[0,t)}) \otimes 1^t$$

$$A_g^+(t) = a(g\chi_{[0,t)}) \otimes 1^t$$

$$\Lambda_\pi(t) = \lambda(\pi\chi_{[0,t)}) \otimes 1^t$$

$t \in \mathbb{R}_+$, where χ_I is the indicator function of the Borel set $I \subseteq \mathbb{R}_+$. The maps A_f, A_g^+, and Λ_π from $\mathbb{R}_+$ to $\tilde{\mathcal{A}}$ are evidently adapted to the filtration $\{\tilde{\mathcal{A}}_t: t \in \mathbb{R}_+\}$ of $\tilde{\mathcal{A}}$.

Let $p, q, u, v \in L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})$, $f, g \in L^\infty_{\Upsilon,\mathrm{loc}}(\mathbb{R}_+)$, and $\pi \in L^\infty_{B(\Upsilon),\mathrm{loc}}(\mathbb{R}_+)$. In the sequel, we interpret the stochastic integral

$$\int_{t_0}^t (p(s)\, d\Lambda_\pi(s) + q(s)\, dA_f(s) + u(s)\, dA_g^+(s) + v(s)\, ds)$$

$(t_0, t) \in \mathbb{R}^2_+$ with $t_0 < t$, as in Hudson and Parthasarathy (1984).

5.1. Stochastic Inclusions

A map $\Phi: \mathbb{R}_+ \to 2^{\tilde{\mathcal{A}}}$, with closed values, will be called a *multivalued stochastic process* indexed by $\mathbb{R}_+$. Such a process is *adapted* (to the filtration $\{\tilde{\mathcal{A}}_t: t \in \mathbb{R}_+\}$ of $\tilde{\mathcal{A}}$) in case $\Phi(t) \subseteq \tilde{\mathcal{A}}_t$, for each $t \in \mathbb{R}_+$. When Φ is adapted and the map $t \mapsto \|\Phi(t)\|_{\eta,\xi}$, $t \in \mathbb{R}_+$ [see p. 2006 of Ekhaguere (1992) for the definition of $\|\mathcal{M}\|_{\eta,\xi}$ for $\mathcal{M} \subseteq \tilde{\mathcal{A}}$], is in $L^2_{\mathrm{loc}}(\mathbb{R}_+)$ for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, then Φ is called *locally absolutely square integrable*. The notation $L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})_{\mathrm{mvs}}$ denotes the set of all locally absolutely square-integrable multivalued stochastic processes on $\mathbb{R}_+$, and we write $L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \tilde{\mathcal{A}})_{\mathrm{mvs}}$ for the set of all multifunctions $\Phi: \mathbb{R}_+ \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$, with closed values, such that the map $t \mapsto \Phi(t, X(t))$, $t \in \mathbb{R}_+$, is in $L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})_{\mathrm{mvs}}$ for every $X \in L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})$.

For $\Phi \in L^2_{\mathrm{loc}}(\mathbb{R}_+ \times \tilde{\mathcal{A}})_{\mathrm{mvs}}$, let

$$L_2(\Phi) \equiv \{\varphi \in L^2_{\text{loc}}(\mathbb{R}_+ \times \tilde{\mathcal{A}}): \varphi \text{ is a selection of } \Phi\}$$

Then, if $\Phi \in L^2_{\text{loc}}(\mathbb{R}_+ \times \tilde{\mathcal{A}})_{\text{mvs}}$, $X \in L^2_{\text{loc}}(\tilde{\mathcal{A}})$ and M denotes any of the stochastic processes A_f, A_g^+, Λ_π and $s \mapsto s\mathbb{1}$, $s \in \mathbb{R}_+$, we define the stochastic integral of $\Phi(\cdot, X(\cdot))$ with respect to M by

$$\int_{t_0}^t \Phi(s, X(s))\, dM(s) \equiv \left\{\int_{t_0}^t \varphi(s, X(s))\, dM(s): \varphi \in L_2(\Phi)\right\}$$

t_0, $t \in \mathbb{R}_+$. This leads to the notion of a *quantum stochastic integral* [resp. *differential*] *inclusion*, as introduced in Ekhaguere (1992).

In the sequel, E, F, G, H lie in $L^2_{\text{loc}}(\mathbb{R}_+ \times \tilde{\mathcal{A}})_{\text{mvs}}$ and we are concerned with the following initial value stochastic differential inclusion:

$$\begin{aligned} dX(t) \in -(&E(t, X(t))\, d\Lambda_\pi(t) + F(t, X(t))\, dA_f(t) + G(t, X(t))\, dA_g^+(t) \\ &+ H(t, X(t))\, dt), \qquad \text{almost all} \quad t \in \mathbb{R}_+ \qquad (5.1)_0 \\ X(0) &= x_0 \qquad \text{for some} \quad x_0 \in \tilde{\mathcal{A}} \end{aligned}$$

This inclusion may be recast as follows. For α, $\beta \in L^\infty_{\Upsilon,\text{loc}}(\mathbb{R})$, define the multifunction $P_{\alpha\beta}$: $\mathbb{R}_+ \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ by

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x)$$

where $\mu_{\alpha\beta}(t) = \langle\alpha(t), \pi(t)\beta(t)\rangle_\Upsilon$, $\nu_\beta(t) = \langle f(t), \beta(t)\rangle_\Upsilon$, and $\sigma_\alpha(t) = \langle\alpha(t), g(t)\rangle_\Upsilon$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, and $\langle\cdot, \cdot\rangle_\Upsilon$ is the inner product of the Hilbert space Υ. This gives rise to the multifunction

$$P: \quad \mathbb{R}_+ \times \tilde{\mathcal{A}} \to 2^{\text{sesq}(\mathbb{D}\underline{\otimes}\mathbb{E})}$$

defined by

$$P(t, x)(\eta, \xi) = \langle\eta, P_{\alpha\beta}(t, x)\xi\rangle = \{\langle\eta, p_{\alpha\beta}(t, x)\xi\rangle: p_{\alpha\beta}(t, x) \in P_{\alpha\beta}(t, x)\} \tag{5.2}$$

$(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, α, $\beta \in L^\infty_{\Upsilon,\text{loc}}(\mathbb{R}_+)$, u, $v \in \mathbb{D}$. Then, by Theorem 6.2 of Ekhaguere (1992), the initial value stochastic differential inclusion $(5.1)_0$ is equivalent to the following initial value nonclassical differential inclusion:

$$\begin{aligned} \frac{d}{dt}\langle\eta, X(t)\xi\rangle &\in -P(t, X(t))(\eta, \xi), \qquad \text{almost all} \quad t \in \mathbb{R}_+ \qquad (5.1)_P \\ X(0) &= x_0 \in \tilde{\mathcal{A}} \end{aligned}$$

for arbitrary $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$.

Definition 5.1. A map $\varphi\colon \mathbb{R}_+ \to \tilde{\mathcal{A}}$ is a *solution* of Problem $(5.1)_0$ if it is weakly absolutely continuous and

$$d\varphi(t) \in -(E(t, \varphi(t))\, d\Lambda_\pi(t) + F(t, \varphi(t))\, dA_f(t) + G(t, \varphi(t))\, dA_g^+(t) + H(t, \varphi(t))dt), \qquad \text{almost all} \quad t \in \mathbb{R}_+$$

$$\varphi(0) = x_0 \in \tilde{\mathcal{A}}$$

Remark. 1. For accounts of the theory of classical differential inclusions, see Aubin and Cellina (1984), Deimling (1992), and Kisielewicz (1991).

2. Notice that Problem $(5.1)_P$ presents Problem $(5.1)_0$ as a nonclassical differential inclusion of *nonlinear evolution type*.

3. The subsequent discussion is concerned with the problem of the existence and uniqueness of a solution of Problem $(5.1)_0$ [or equivalently Problem $(5.1)_P$], under a monotonicity condition on P.

Definition 5.2. Problem $(5.1)_0$ will be said to be of *hypermaximal monotone type* if the multifunction P in $(5.1)_0$ is such that $\mathcal{P} = P \otimes \mathbb{1}$ lies in $\text{Hypmax}(\mathbb{R}_+ \times \tilde{\mathcal{A}})$. Similarly, Problem $(5.1)_0$ is *Lipschitzian* if P is Lipschitzian, as explained in Ekhaguere (1992).

Remark. 1. In Ekhaguere (1992) we established the existence of a solution of a *Lipschitzian* stochastic differential inclusion, and proved a *relaxation theorem* giving the relationship between the solutions of such an inclusion and those of its convexification.

2. The main result of this section is Theorem 5.6. It establishes the existence of a *unique* adapted solution of a stochastic differential inclusion of hypermaximal monotone type. This solution is arrived at by a limiting process involving the unique adapted solutions of a one-parameter family of *Lipschitzian* stochastic differential *equations*. We shall first introduce these equations.

5.2. The Approximating Lipschitzian Equations

Let P in $(5.1)_P$ be such that $\mathcal{P} = P \otimes \mathbb{1}$ is in $\text{Hypmax}(\mathbb{R}_+ \times \tilde{\mathcal{A}})$. Let $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, $\lambda > 0$, and $\alpha, \beta \in L^\infty_{\Upsilon,\text{loc}}(\mathbb{R})$. Using

$$P_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E(t, x) + \nu_\beta(t)F(t, x) + \sigma_\alpha(t)G(t, x) + H(t, x)$$

we have

$$P_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t, x)) = \mu_{\alpha\beta}(t)E(t, J_{\lambda,\alpha\beta}(t, x)) + \nu_\beta(t)F(t, J_{\lambda,\alpha\beta}(t, x)) + \sigma_\alpha(t)G(t, J_{\lambda,\alpha\beta}(t, x)) + H(t, J_{\lambda,\alpha\beta}(t, x))$$

Then, since $P_{\lambda,\alpha\beta}(t, x) \in P_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t, x))$ by Theorem 4.1(1)(iii), it follows that there are $E_{\lambda,\alpha\beta}(t, x) \in E(t, J_{\lambda,\alpha\beta}(t, x))$, $F_{\lambda,\alpha\beta}(t, x) \in F(t, J_{\lambda,\alpha\beta}(t, x))$, $G_{\lambda,\alpha\beta}(t, x) \in G(t, J_{\lambda,\alpha\beta}(t, x))$, and $H_{\lambda,\alpha\beta}(t, x) \in H(t, J_{\lambda,\alpha\beta}(t, x))$ such that

$$\begin{aligned} P_{\lambda,\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)E_{\lambda,\alpha\beta}(t, x) &+ \nu_{\beta}(t)F_{\lambda,\alpha\beta}(t, x) \\ &+ \sigma_{\alpha}(t)G_{\lambda,\alpha\beta}(t, x) + H_{\lambda,\alpha\beta}(t, x) \end{aligned}$$

For $Q \in \{P, E, F, G, H\}$, define the quadratic form $Q_{\lambda}(t, x)$ by

$$Q_{\lambda}(t, x)(\eta, \xi) = \langle \eta, Q_{\lambda,\alpha\beta}(t, x)\xi \rangle$$

for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, and introduce the initial value stochastic differential *equation*:

$$\begin{aligned} dX_{\lambda}(t) = -(E_{\lambda}(t, X_{\lambda}(t))\, d\Lambda_{\pi}(t) &+ F_{\lambda}(t, X_{\lambda}(t))\, dA_f(t) + G_{\lambda}(t, X_{\lambda}(t))\, dA_g^{+}(t) \\ &+ H_{\lambda}(t, X_{\lambda}(t))\, dt), \qquad \text{almost all} \quad t \in \mathbb{R}_+ \qquad (5.1)_{\lambda} \\ X_{\lambda}(0) = x_0 \in \tilde{\mathcal{A}} & \end{aligned}$$

This equation is equivalent to the initial value differential *equation*

$$\begin{aligned} \frac{d}{dt}\langle \eta, X_{\lambda}(t)\xi \rangle &= -P_{\lambda}(t, X_{\lambda}(t))(\eta, \xi), \qquad \text{almost all} \quad t \in \mathbb{R}_+ \quad (5.1)_{P_{\lambda}} \\ X_{\lambda}(0) &= x_0 \in \tilde{\mathcal{A}} \end{aligned}$$

for all $(\eta, \xi) \in (\mathbb{D} \underline{\otimes} \mathbb{E})^2$, where P_{λ}, $\lambda > 0$, is the *Yosida approximation* of P described in Section 4.

As P_{λ} is Lipschitzian for each $\lambda > 0$, Problem $(5.1)_{\lambda}$ has a *unique* adapted solution which is arrived at by Picard's method of successive approximation.

5.3. The Main Result

Throughout the rest of the discussion, *we assume that* Problem $(5.1)_0$, *or equivalently* Problem $(5.1)_P$, *is of hypermaximal monotone type.*

Notation. Let $I = [T_0, T)$, with $T > T_0 \geq 0$. Then, $C(I, \tilde{\mathcal{A}})$ is the locally convex space of maps: $\varphi: I \to \tilde{\mathcal{A}}$ whose topology is generated by the family $\{\|\cdot\|_{\mathrm{con},\eta\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ of seminorms defined by

$$\|\varphi\|_{\mathrm{con},\eta\xi} = \sup_{t\in I}\|\varphi(t)\|_{\eta,\xi}, \qquad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

We shall show that for each compact subinterval $I \subseteq \mathbb{R}_+$, a solution of Problem $(5.1)_{\lambda}$ converges in $C(I, \tilde{\mathcal{A}})$ to a solution of Problem $(5.1)_0$ as $\lambda \downarrow 0$. This will be done in stages as follows.

Proposition 5.3. If φ_1 and φ_2 are two solutions of Problem $(5.1)_0$ satisfying $\varphi_1(0) = x_{10}$ and $\varphi_2(0) = x_{20}$, for some $x_{10}, x_{20} \in \tilde{\mathcal{A}}$, then

$$\|\varphi_1(t) - \varphi_2(t)\|_{\eta,\xi} \leq \|x_{10} - x_{20}\|_{\eta,\xi}$$

for all $t \in \mathbb{R}_+$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Proof. Let φ denote either φ_1 or φ_2. As φ is a solution of Problem $(5.1)_0$, there are single-valued maps ω_E, ω_F, ω_G, ω_H of $\mathbb{R}_+ \times \tilde{\mathcal{A}}$ into $\tilde{\mathcal{A}}$ such that $\omega_E(t, x) \in E(t, x)$, $\omega_F(t, x) \in F(t, x)$, $\omega_G(t, x) \in G(t, x)$, $\omega_H(t, x) \in H(t, x)$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, and

$$d\varphi(t) = -(\omega_E(t, \varphi(t))\, d\Lambda_\pi(t) + \omega_F(t, \varphi(t))\, dA_f(t) + \omega_G(t, \varphi(t))\, dA_g^+(t) + \omega_H(t, \varphi(t))\, dt), \qquad \text{almost all} \quad t \in \mathbb{R}_+$$

Then, for $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{Y,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, we have

$$\begin{aligned} d\|\varphi_1(t) - \varphi_2(t)\|^2_{\eta,\xi} &= -2\,\mathrm{Re}(\langle((d\varphi_1(t) - d\varphi_2(t)) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\varphi_1(t), \varphi_2(t))\rangle_{(2)})\, dt \\ &= -2\,\mathrm{Re}(\langle((p_{\alpha\beta}(t, \varphi_1(t)) - p_{\alpha\beta}(t, \varphi_2(t))) \otimes 1)(\xi \otimes \eta), \\ &\quad \Phi_{(\eta,\xi)}(\varphi_1(t), \varphi_2(t))\rangle_{(2)})\, dt \end{aligned}$$

where

$$p_{\alpha\beta}(t, \varphi(t)) = \mu_{\alpha\beta}(t)\omega_E(t, \varphi(t)) + \nu_\beta(t)\omega_F(t, \varphi(t)) + \sigma_\alpha(t)\omega_G(t, \varphi(t)) + \omega_H(t, \varphi(t))$$

$t \in \mathbb{R}_+$. As $p_{\alpha\beta}(t, \varphi(t)) \in P_{\alpha\beta}(t, \varphi(t))$, $t \in \mathbb{R}_+$, and $\mathcal{P} = P \otimes 1$ is *monotone*, it follows that

$$d\|\varphi_1(t) - \varphi_2(t)\|^2_{\eta,\xi} \leq 0$$

for all $t \in \mathbb{R}_+$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, whence

$$\|\varphi_1(t) - \varphi_2(t)\|_{\eta,\xi} \leq \|\varphi_1(0) - \varphi_2(0)\|_{\eta,\xi} = \|x_{10} - x_{20}\|_{\eta,\xi}$$

for all $t \in \mathbb{R}_+$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. This concludes the proof. ■

Remark. We have seen in Theorem 4.1(1)(i) that the *resolvent* J_λ of P has the property that $x \mapsto J_{\lambda,\alpha\beta}(t, x)$, $x \in \tilde{\mathcal{A}}$, is *Lipschitzian*, with Lipschitz constant 1, for each $t \in \mathbb{R}_+$, $\lambda > 0$, and $\alpha, \beta \in L^\infty_{Y,\mathrm{loc}}(\mathbb{R}_+)$. To discuss the convergence of the net $\{\varphi_\lambda: \lambda > 0\}$ to a solution of Problem $(5.1)_0$, where φ_λ is a solution of Problem $(5.1)_\lambda$, we require a continuity condition on the single-valued map $t \mapsto J_{\lambda,\alpha\beta}(t, x)$, $t \in \mathbb{R}_+$, for arbitrary $x \in \tilde{\mathcal{A}}$, $\lambda > 0$, and

$\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$. Under the continuity condition, we first establish *a priori* bounds on $\|\varphi_\lambda(t)\|_{\eta,\xi}$ and $|(d/dt)\langle\eta, \varphi_\lambda(t)\xi\rangle|$ that are independent of λ, for each $t \in \mathbb{R}_+$ and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Since we are interested in the limit as $\lambda \downarrow 0$, we can restrict λ to the interval $(0, 1]$, as we do below.

Proposition 5.4. Let $\lambda \in (0, 1]$, φ_λ a solution of $(5.1)_\lambda$, $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, and $s > 0$. Suppose that there are a monotone increasing function $\Psi: \mathbb{R}_+ \to \mathbb{R}_+$ and a collection $\{c_{\eta\xi}: \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ of continuous functions from $\mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|J_{\lambda,\alpha\beta}(t + s, x) - J_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi} \le \lambda c_{\eta\xi}(t)s(\|J_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi} + \Psi(\|x\|_{\eta,\xi}))$$

for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

Then, for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, there are continuous functions $k^{(1)}_{\eta\xi}$, $k^{(2)}_{\eta\xi}: \mathbb{R}_+ \to \mathbb{R}_+$, independent of $\lambda \in (0, 1]$, such that the following estimates hold:

(i) $\|\varphi_\lambda(t)\|_{\eta,\xi} \le k^{(1)}_{\eta\xi}(t)$, for all $\lambda \in (0, 1]$, $t \in \mathbb{R}_+$.
(ii) $|(d/dt)\langle\eta, \varphi_\lambda(t)\xi\rangle| \le k^{(2)}_{\eta\xi}(t)$, for all $t \in \mathbb{R}_+$, $\lambda \in (0, 1]$.

Proof. The estimates employ Gronwall's inequality (Walter, 1964).

(i) We have

$$\begin{aligned}
&\frac{d}{dt}\|\varphi_\lambda(t) - x_0\|^2_{\eta,\xi}\\
&= -2\,\mathrm{Re}(\langle((P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\varphi_\lambda(t), x_0)\rangle_{(2)})\\
&= -2\,\mathrm{Re}(\langle((P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t)) - P_{\lambda,\alpha\beta}(t, x_0)) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\varphi_\lambda(t), x_0)\rangle_{(2)})\\
&\quad -2\,\mathrm{Re}(\langle(P_{\lambda,\alpha\beta}(t, x_0) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\varphi_\lambda(t), x_0)\rangle_{(2)})\\
&\le -2\,\mathrm{Re}(\langle(P_{\lambda,\alpha\beta}(t, x_0) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\varphi_\lambda(t), x_0)\rangle_{(2)})
\end{aligned}$$

as $\mathcal{P}_\lambda = P_\lambda \otimes 1$ is monotone, by Theorem 4.1(1)(iv).

But

$$\begin{aligned}
&-2\,\mathrm{Re}(\langle(P_{\lambda,\alpha\beta}(t, x_0) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\varphi_\lambda(t), x_0)\rangle_{(2)})\\
&\le 2|(\langle(P_{\lambda,\alpha\beta}(t, x_0) \otimes 1)(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\varphi_\lambda(t), x_0)\rangle_{(2)})|\\
&= 2|\langle P_{\lambda,\alpha\beta}(t, x_0)\xi, \eta\rangle| \cdot |\langle\eta, (\varphi_\lambda(t) - x_0)\xi\rangle|\\
&= 2\|P_{\lambda,\alpha\beta}(t, x_0)\|_{\eta,\xi}\|\varphi_\lambda - x_0\|_{\eta,\xi}\\
&\le 2\|m_{\alpha\beta}(t, x_0)\|_{\eta,\xi}\|\varphi_\lambda - x_0\|_{\eta,\xi}\\
&\qquad\qquad [\text{by Theorem 4.1(2)}]\\
&\le \|m_{\alpha\beta}(t, x_0)\|^2_{\eta,\xi} + \|\varphi_\lambda - x_0\|^2_{\eta,\xi}
\end{aligned}$$

Hence

$$\frac{d}{dt}\|\varphi_\lambda(t) - x_0\|^2_{\eta,\xi} \leq \|m_{\alpha\beta}(t, x_0)\|^2_{\eta,\xi} + \|\varphi_\lambda - x_0\|^2_{\eta,\xi}$$

whence, by Gronwall's inequality, we have

$$\|\varphi_\lambda(t) - x_0\|_{\eta,\xi} \leq k^{(0)}_{\eta\xi}(t)$$

where $k^{(0)}_{\eta\xi}$: $\mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, independent of λ. Hence, $\|\varphi_\lambda(t)\|_{\eta,\xi} \leq k^{(1)}_{\eta\xi}(t)$, with $k^{(1)}_{\eta\xi}(t) = \|x_0\|_{\eta,\xi} + k^{(0)}_{\eta\xi}(t)$, for each $t \in \mathbb{R}_+$ and all $\lambda \in (0, 1]$.

(ii) Let $t, s \in \mathbb{R}_+$, $\lambda \in (0, 1]$. Then

$$\begin{aligned}
&\frac{d}{dt}\|\varphi_\lambda(t+s) - \varphi_\lambda(t)\|^2_{\eta,\xi} \\
&\quad = -2\,\mathrm{Re}(\langle((P_{\lambda,\alpha\beta}(t+s, \varphi_\lambda(t+s)) - P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))) \otimes \mathbb{1})(\xi \otimes \eta), \\
&\qquad\quad \Phi_{(\eta,\xi)}(\varphi_\lambda(t+s), \varphi_\lambda(t))\rangle_{(2)}) \\
&\quad = -2\,\mathrm{Re}(\langle((P_{\lambda,\alpha\beta}(t+s, \varphi_\lambda(t+s)) - P_{\lambda,\alpha\beta}(t+s, \varphi_\lambda(t))) \otimes \mathbb{1})(\xi \otimes \eta), \\
&\qquad\quad \Phi_{(\eta,\xi)}(\varphi_\lambda(t+s), \varphi_\lambda(t))\rangle_{(2)}) \\
&\qquad -2\,\mathrm{Re}(\langle((P_{\lambda,\alpha\beta}(t+s, \varphi_\lambda(t)) - P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))) \otimes \mathbb{1})(\xi \otimes \eta), \\
&\qquad\quad \Phi_{(\eta,\xi)}(\varphi_\lambda(t+s), \varphi_\lambda(t))\rangle_{(2)}) \\
&\quad \leq -2\,\mathrm{Re}(\langle((P_{\lambda,\alpha\beta}(t+s, \varphi_\lambda(t)) - P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))) \otimes \mathbb{1})(\xi \otimes \eta), \\
&\qquad\quad \Phi_{(\eta,\xi)}(\varphi_\lambda(t+s), \varphi_\lambda(t))\rangle_{(2)}) \\
&\qquad\qquad [\text{since } \mathcal{P}_\lambda = P_\lambda \otimes \mathbb{1} \text{ is monotone, by Theorem 4.1(1)(iv)}] \\
&\quad \leq 2\|P_{\lambda,\alpha\beta}(t+s, \varphi_\lambda(t)) - P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))\|_{\eta,\xi}\|\varphi_\lambda(t+s) - \varphi_\lambda(t)\|_{\eta,\xi}
\end{aligned}$$

Appealing to the inequality of the proposition, we have

$$\begin{aligned}
&\|P_{\lambda,\alpha\beta}(t+s, \varphi_\lambda(t)) - P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))\|_{\eta,\xi} \\
&\quad = \frac{1}{\lambda}\|J_{\lambda,\alpha\beta}(t+s, \varphi_\lambda(t)) - J_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))\|_{\eta,\xi} \\
&\quad \leq c_{\eta\xi}(t)s(\|J_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))\|_{\eta,\xi} + \Psi(\|\varphi_\lambda(t)\|_{\eta,\xi})) \\
&\quad = c_{\eta\xi}(t)s(\|\varphi_\lambda(t) + \lambda P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))\|_{\eta,\xi} + \Psi(\|\varphi_\lambda(t)\|_{\eta,\xi})) \\
&\quad \leq c_{\eta\xi}(t)s(\|\varphi_\lambda(t)\|_{\eta,\xi} + \|P_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))\|_{\eta,\xi} + \Psi(\|\varphi_\lambda(t)\|_{\eta,\xi})) \\
&\quad \leq c_{\eta\xi}(t)s\left(k^{(3)}_{\eta\xi}(t) + \left|\frac{d}{dt}\langle\eta, \varphi_\lambda(t)\xi\rangle\right|\right)
\end{aligned}$$

by (i) above, where $k^{(3)}_{\eta\xi}$ is the continuous function on $\mathbb{R}_+$ given by $k^{(3)}_{\eta\xi}(t) = k^{(1)}_{\eta\xi}(t) + \Psi(k^{(1)}_{\eta\xi}(t))$, and since φ_λ is a solution of $(5.1)_\lambda$.

Hence,

$$\begin{aligned}
&\frac{d}{dt}\|\varphi_\lambda(t+s) - \varphi_\lambda(t)\|^2_{\eta,\xi} \\
&\quad \le 2c_{\eta\xi}(t)s\left(k^{(3)}_{\eta\xi}(t) + \left|\frac{d}{dt}\langle\eta, \varphi_\lambda(t)\xi\rangle\right|\right)\|\varphi_\lambda(t+s) - \varphi_\lambda(t)\|_{\eta,\xi} \\
&\quad \le s^2\left(k^{(3)}_{\eta\xi}(t) + \left|\frac{d}{dt}\langle\eta, \varphi_\lambda(t)\xi\rangle\right|\right)^2 + (c_{\eta\xi}(t))^2\|\varphi_\lambda(t+s) - \varphi_\lambda(t)\|^2_{\eta,\xi}
\end{aligned}$$

whence

$$\begin{aligned}
&\|\varphi_\lambda(t+s) - \varphi_\lambda(t)\|^2_{\eta,\xi} \\
&\quad \le \|\varphi_\lambda(s) - \varphi_\lambda(0)\|^2_{\eta,\xi}\exp[C_{\eta\xi}(t)] \\
&\qquad + 2s^2\int_0^t dr\exp[C_{\eta\xi}(t) - C_{\eta\xi}(r)]\left([k^{(3)}_{\eta\xi}(t)]^2 + \left|\frac{d}{dr}\langle\eta, \varphi_\lambda(r)\xi\rangle\right|^2\right)
\end{aligned}$$

by Gronwall's inequality, where $C_{\eta\xi}(t) = \int_0^t [c_{\eta\xi}(r)]^2$. Dividing both sides of the last inequality by s^2 and letting $s \downarrow 0$, we get

$$\begin{aligned}
&\left|\frac{d}{dt}\langle\eta, \varphi_\lambda(t)\xi\rangle\right|^2 \\
&\quad \le \left|\left.\frac{d}{dt}\langle\eta, \varphi_\lambda(t)\xi\rangle\right|_{t=0}\right|^2 \exp[C_{\eta\xi}(t)] \\
&\qquad + 2\int_0^t dr\exp[C_{\eta\xi}(t) - C_{\eta\xi}(r)]\left([k^{(3)}_{\eta\xi}(t)]^2 + \left|\frac{d}{dr}\langle\eta, \varphi_\lambda(r)\xi\rangle\right|^2\right) \\
&\quad \le \|m_{\alpha\beta}(0, x_0)\|^2_{\eta,\xi}\exp[C_{\eta\xi}(t)] \\
&\qquad + 2\int_0^t dr\exp[C_{\eta\xi}(t) - C_{\eta\xi}(r)]\left([k^{(3)}_{\eta\xi}(t)]^2 + \left|\frac{d}{dr}\langle\eta, \varphi_\lambda(r)\xi\rangle\right|^2\right) \\
&\qquad\quad \left[\text{since } \left|\left.\frac{d}{dt}\langle\eta, \varphi_\lambda(t)\rangle\right|_{t=0}\right|^2 = \|P_{\lambda,\alpha\beta}(0, \varphi_\lambda(0))\|^2_{\eta,\xi} \le \|m_{\alpha\beta}(0, x_0)\|^2_{\eta,\xi}\right. \\
&\qquad\quad \left.\text{by Theorem 4.1(2)}\right] \\
&\quad = k^{(4)}_{\eta\xi}(t) + \int_0^t dr\, k^{(5)}_{\eta\xi}(t)\left|\frac{d}{dr}\langle\eta, \varphi_\lambda(r)\xi\rangle\right|^2
\end{aligned}$$

where $k^{(4)}_{\eta\xi}$, $k^{(5)}_{\eta\xi}$: $\mathbb{R}_+ \to \mathbb{R}_+$ are continuous with obvious definitions. Then, by Gronwall's inequality, we finally have

$$\left|\frac{d}{dt}\langle\eta, \varphi_\lambda(t)\xi\rangle\right| \leq k^{(2)}_{\eta\xi}(t), \qquad t \in \mathbb{R}_+$$

for some continuous function $k^{(2)}_{\eta\xi}$: $\mathbb{R}_+ \to \mathbb{R}_+$ independent of $\lambda \in (0, 1]$. This concludes the proof. ∎

Remark. We look next at the issue of convergence of the net $\{\varphi_\lambda: \lambda \in (0, 1]\}$, where φ_λ is a solution of $(5.1)_\lambda$.

Notation. Let T_0, T be arbitrary, with $T > T_0 \geq 0$ and $I = [T_0, T)$. Then, with $k^{(2)}_{\eta\xi}$, $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, as in Proposition 5.4(ii), we define the number $k^{(2)}_{T_0,T,\eta\xi}$ by

$$k^{(2)}_{T_0,T,\eta\xi} = \sup_{t\in I=[T_0,T)} k^{(2)}_{\eta\xi}(t), \qquad \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$$

Proposition 5.5. Let φ_λ be the unique adapted solution of Problem $(5.1)_\lambda$, $\lambda > 0$. Then, for each compact subinterval $I = [T_0, T) \subseteq \mathbb{R}_+$, with $T > T_0 \geq 0$, the family $\{\varphi_\lambda: \lambda \in (0, 1]\}$ is a Cauchy net in $C(I, \tilde{\mathcal{A}})$ which converges on I to a weakly absolutely continuous adapted member φ of $C(I, \tilde{\mathcal{A}})$.

Proof. Let $\lambda \in (0, 1]$ and $I = [T_0, T)$. Then, by Proposition 5.4(ii), we have

$$|P_\lambda(t, \varphi_\lambda(t))(\eta, \xi)| = \left|\frac{d}{dt}\langle\eta, \varphi_\lambda(t)\xi\rangle\right| \leq k^{(2)}_{\eta\xi}(t) \leq k^{(2)}_{T_0,T,\eta\xi}$$

for all $(\lambda, t) \in (0, 1] \times I$ and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\mathbb{Y},\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. This shows that $\{P_\lambda(t, \varphi_\lambda(t))(\eta, \xi): \lambda \in (0, 1]\}$ is a bounded net of complex numbers for arbitrary $t \in I$ and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$.

Next, for $\lambda, \mu \in (0, 1]$, $t \in I$, and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{\mathbb{Y},\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, we have

$$\begin{aligned}
&\frac{1}{2}\|\varphi_\lambda(t) - \varphi_\mu(t)\|^2_{\eta,\xi} \\
&= \frac{1}{2}\int_0^t d\|\varphi_\lambda(s) - \varphi_\mu(s)\|^2_{\eta,\xi} \\
&= \int_0^t \mathrm{Re}(\langle(d\varphi_\lambda(s) - d\varphi_\mu(s))\xi, \eta\rangle\langle\eta, (\varphi_\lambda(s) - \varphi_\mu(s))\xi\rangle) \\
&= -\int_0^t ds\, \mathrm{Re}(\langle(P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s)) - P_{\mu,\alpha\beta}(s, \varphi_\mu(s)))\xi, \eta\rangle\langle\eta, (\varphi_\lambda(s) - \varphi_\mu(s))\xi\rangle)
\end{aligned}$$

$$= -\int_0^t ds \operatorname{Re}(\langle((P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s)) - P_{\mu,\alpha\beta}(s, \varphi_\mu(s))) \otimes 1)(\xi \otimes \eta),$$

$$\Phi_{(\eta,\xi)}(\varphi_\lambda(s), \varphi_\mu(s))\rangle_{(2)})$$

$$= -\int_0^t ds \operatorname{Re}\{\langle((P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s)) - P_{\mu,\alpha\beta}(s, \varphi_\mu(s))) \otimes \mathbb{1})(\xi \otimes \eta),$$

$$\Phi_{(\eta,\xi)}(\lambda P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s)), \mu P_{\mu,\alpha\beta}(s, \varphi_\mu(s)))\rangle_{(2)}$$

$$+ \langle((P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s)) - P_{\mu,\alpha\beta}(s, \varphi_\mu(s))) \otimes \mathbb{1})(\xi \otimes \eta),$$

$$\Phi_{(\eta,\xi)}(J_{\lambda,\alpha\beta}(s, \varphi_\lambda(s)), J_{\mu,\alpha\beta}(s, \varphi_\mu(s)))\rangle_{(2)}\}$$

[by using $x = J_{\sigma,\alpha\beta}(t, x) + \sigma P_{\sigma,\alpha\beta}(t, x)$, $\sigma > 0$]

$$\leq -\int_0^t ds \operatorname{Re}\{\langle((P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s)) - P_{\mu,\alpha\beta}(s, \varphi_\mu(s))) \otimes \mathbb{1})(\xi \otimes \eta),$$

$$\Phi_{(\eta,\xi)}(\lambda P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s)), \mu P_{\mu,\alpha\beta}(s, \varphi_\mu(s)))\rangle_{(2)}\}$$

[since $\mathcal{P} = P \otimes \mathbb{1}$ is monotone and 1(iii) of Theorem (4.1) holds]

$$= \int_0^t ds \operatorname{Re}\{\lambda\langle P_{\mu,\alpha\beta}(s, \varphi_\mu(s)), \eta\rangle\langle\eta, P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\rangle$$

$$+ \mu\langle P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\xi, \eta\rangle\langle\eta, P_{\mu,\alpha\beta}(s, \varphi_\mu(s))\xi\rangle\}$$

$$- \int_0^t ds\, (\lambda|\langle\eta, P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\xi\rangle|^2 + \mu|\langle\eta, P_{\mu,\alpha\beta}(s, \varphi_\mu(s))\xi\rangle|^2)$$

As

$$|\lambda \operatorname{Re}(\langle P_{\mu,\alpha\beta}(s, \varphi_\mu(s))\xi, \eta\rangle\langle\eta, P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\xi\rangle)|$$

$$\leq \lambda|\langle P_{\mu,\alpha\beta}(s, \varphi_\mu(s))\xi, \eta\rangle||\langle\eta, P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\xi\rangle|$$

$$\leq \frac{\lambda}{4}|\langle\eta, P_{\mu,\alpha\beta}(s, \varphi_\mu(s))\xi\rangle|^2 + \lambda|\langle\eta, P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\xi\rangle|^2$$

and (similarly)

$$|\mu \operatorname{Re}(\langle P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\xi, \eta\rangle\langle\eta, P_{\mu,\alpha\beta}(s, \varphi_\mu(s))\xi\rangle)|$$

$$\leq \frac{\mu}{4}|\langle\eta, P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\xi\rangle|^2 + \mu|\langle\eta, P_{\mu,\alpha\beta}(s, \varphi_\mu(s))\xi\rangle|^2$$

it follows that

$$\frac{1}{2}\|\varphi_\lambda(t) - \varphi_\mu(t)\|^2_{\eta,\xi}$$

$$\le \frac{1}{4}\int_0^t ds\,(\lambda|\langle\eta, P_{\mu,\alpha\beta}(s, \varphi_\mu(s))\xi\rangle|^2 + \mu|\langle\eta, P_{\lambda,\alpha\beta}(s, \varphi_\lambda(s))\rangle|^2)$$

$$\le \frac{t}{4}(\lambda + \mu)(k^{(2)}_{T_0,T,\eta\xi})^2$$

for all $t \in I$, λ, $\mu \in (0, 1]$, η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, showing that the net $\{\varphi_\lambda: \lambda \in (0, 1]\}$ is Cauchy in $C(I, \tilde{\mathscr{A}})$ and converges to some φ in $C(I, \tilde{\mathscr{A}})$ as $\lambda \downarrow 0$. This φ is weakly absolutely continuous. To see this, let $T_0 \le t_0 < t_1 < \cdots < t_n = T$ be a disjoint partition of $[T_0, T)$, with $\Sigma_{j=0}^{n-1}(t_{j+1} - t_j) < \infty$, and η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, α, $\beta \in L^\infty_{\mathrm{Y,loc}}(\mathbb{R}_+)$, u, $v \in \mathbb{D}$. Then, from

$$\left|\frac{d}{ds}\langle\eta, \varphi_\lambda(s)\xi\rangle\right| = |P_\lambda(s, \varphi_\lambda(s))(\eta, \xi)| \le k^{(2)}_{T_0,T,\eta\xi}$$

we get

$$|\langle\eta, \varphi_\lambda(t_{j+1})\xi\rangle - \langle\eta, \varphi_\lambda(t_j)\xi\rangle|$$

$$= \left|\int_{t_j}^{t_{j+1}} ds\,\frac{d}{ds}\langle\eta, \varphi_\lambda(s)\xi\rangle\right|$$

$$\le \int_{t_j}^{t_{j+1}} ds\left|\frac{d}{ds}\langle\eta, \varphi_\lambda(s)\xi\rangle\right|$$

$$\le k^{(2)}_{T_0,T,\eta\xi}(t_{j+1} - t_j)$$

Hence, letting $\lambda \downarrow 0$ and summing over j, we get

$$\sum_{j=0}^{n-1}|\langle\eta, \varphi(t_{j+1})\xi\rangle - \langle\eta, \varphi(t_j)\xi\rangle| \le k^{(2)}_{T_0,T,\eta\xi}\sum_{j=0}^{n-1}(t_{j+1} - t_j)$$

showing that φ is weakly absolutely continuous. Finally, it is clear that φ is adapted.

This concludes the proof. ■

Remark. As $t \mapsto \langle\eta, \varphi(t)\xi\rangle$, $t \in \mathbb{R}_+$, is absolutely continuous for arbitrary η, $\xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, there is a set of Lebesgue measure zero in $\mathbb{R}_+$ outside which it is differentiable (Hewitt and Stromberg, 1965), showing that φ is weakly differentiable at almost every $t \in \mathbb{R}_+$.

Notation. In connection with the proof of our main theorem below, we employ the following notation.

1. Let $T > 0$ and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ be arbitrary, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

We write $L^2_{\eta\xi}([0, T))$ for the closure in $L^2([0, T))$ of the linear span of the set $\{t \mapsto \langle \eta, \theta(t)\xi \rangle, t \in [0, T): \theta \in L^2([0, T), \tilde{\mathcal{A}})\}$ of complex-valued functions on $[0, T)$. Then, $L^2_{\eta\xi}([0, T))$ is a Hilbert subspace of $L^2([0, T))$.

2. For $\theta \in L^2([0, T), \tilde{\mathcal{A}})$, we denote the function $t \mapsto \langle \eta, \theta(t)\xi \rangle$, $t \in [0, T)$, by $\theta(\cdot)(\eta, \xi)$ and the multifunction [resp. the single-valued functions] $t \mapsto P(t, \theta(t))(\eta, \xi)$ [resp. $t \mapsto P_\lambda(t, \theta(t))(\eta, \xi)$ and $t \mapsto J_\lambda(t, \theta(t))(\eta, \xi)$] of $[0, T)$ into $2^{\mathbb{C}}$ [resp. into $\mathbb{C}$] by $P(\cdot, \theta(\cdot))(\eta, \xi)$ [resp. $P_\lambda(\cdot, \theta(\cdot))(\eta, \xi)$ and $J_\lambda(\cdot, \theta(\cdot))(\eta, \xi)$]. In general, if p: $[0, T) \to \mathrm{sesq}(\mathbb{D} \underline{\otimes} \mathbb{E})$, we denote the function $t \mapsto p(t)(\eta, \xi)$, $t \in [0, T)$, by $p(\cdot)(\eta, \xi)$.

Remark. The following is our main result.

Theorem 5.6. Suppose that Problem $(5.1)_0$ is of hypermaximal monotone type and the inequality of Proposition 5.4 holds. Then, $(5.1)_0$ possesses a unique adapted solution.

Proof. The issue of uniqueness is settled by Proposition 5.3, since if φ_1 and φ_2 are two solutions of Problem $(5.1)_0$ with the same initial condition, i.e., $\varphi_1(0) = x_0 = \varphi_2(0)$, then

$$\|\varphi_1(t) - \varphi_2(t)\|_{\eta,\xi} = 0$$

for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, $t \in \mathbb{R}_+$, showing that $\varphi_1 = \varphi_2$.

Concerning the issue of existence, we shall show that the adapted map φ in Proposition 5.5 is a solution of Problem $(5.1)_0$.

Let $\lambda \in (0, 1]$, $I = [0, T)$, and φ_λ be as in Proposition 5.5. As

$$\begin{aligned} &\left| \|\varphi_\lambda(t) - \varphi(t)\|_{\eta,\xi} - \|\varphi(t) - J_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))\|_{\eta,\xi} \right| \\ &\leq \|\varphi_\lambda(t) - J_{\lambda,\alpha\beta}(t, \varphi_\lambda(t))\|_{\eta,\xi} \\ &\leq \lambda k^{(2)}_{0,T,\eta\xi} \end{aligned}$$

for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$, it follows that the net $\{J_\lambda(\cdot, \varphi_\lambda(\cdot))(\eta, \xi): \lambda \in (0, 1]\}$ converges in $C([0, T), \tilde{\mathcal{A}})$ to the function $\langle \eta, \varphi(\cdot)\xi \rangle$ as $\lambda \downarrow 0$, for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$. Hence, the net $\{J_\lambda(\cdot, \varphi_\lambda(\cdot))(\eta, \xi): \lambda \in (0, 1]\}$ converges in $L^2_{\eta\xi}([0, T))$ to $\langle \eta, \varphi(\cdot)\xi \rangle$ as $\lambda \downarrow 0$.

Define the multifunction $\tilde{P}$: $L^2_{\eta\xi}([0, T)) \subset L^2([0, T)) \to 2^{L^2([0,T))}$ by

$$\tilde{P}(\theta(\cdot)(\eta, \xi)) = P(\cdot, \theta(\cdot))(\eta, \xi), \qquad \theta \in L^2([0, T), \tilde{\mathcal{A}})$$

Then, $\tilde{P}$ is *monotone*. This claim will be true if

$$\mathrm{Re}(\langle p(\theta(\cdot)(\eta, \xi)) - q(\phi(\cdot)(\eta, \xi)), \theta(\cdot)(\eta, \xi) - \phi(\cdot)(\eta, \xi)\rangle_{L^2([0,T))}) \geq 0$$

whenever $\theta(\cdot)(\eta, \xi), \phi(\cdot)(\eta, \xi) \in L^2_{\eta\xi}([0, T))$ and $p(\theta(\cdot)(\eta, \xi)) \in \tilde{P}(\theta(\cdot)(\eta, \xi))$, $q(\phi(\cdot)(\eta, \xi)) \in \tilde{P}(\phi(\cdot)(\eta, \xi))$. Indeed, this is the case since

$$\begin{aligned}
&\mathrm{Re}(\langle p(\theta(\cdot)(\eta, \xi)) - q(\phi(\cdot)(\eta, \xi)), \theta(\cdot)(\eta, \xi) - \phi(\cdot)(\eta, \xi)\rangle_{L^2([0,T))}) \\
&= \mathrm{Re} \int_0^T \overline{(p(\theta(s)(\eta, \xi)) - q(\phi(s)(\eta, \xi)))}(\theta(s)(\eta, \xi) - \phi(s)(\eta, \xi)) \\
&= \mathrm{Re} \int_0^T ds\, \overline{\langle \eta, (p_{\alpha\beta}(s, \theta(s)) - q_{\alpha\beta}(s, \phi(s)))\xi\rangle}\langle \eta, (\theta(s) - \phi(s))\xi\rangle
\end{aligned}$$

[because $p(\psi(s))(\eta, \xi) \in \tilde{P}(\psi(s)(\eta, \xi)) = P(s, \psi(s))(\eta, \xi)$ is of the form $p(\psi(s))(\eta, \xi) = \langle \eta, p_{\alpha\beta}(s, \psi(s))\xi\rangle$, for some $p_{\alpha\beta}(s, \psi(s)) \in P_{\alpha\beta}(s, \psi(s))$, $s \in \mathbb{R}_+$]

$$\begin{aligned}
&= \mathrm{Re} \int_0^T ds\, \langle (p_{\alpha\beta}(s, \theta(s)) - q_{\alpha\beta}(s, \phi(s)))\xi, \eta\rangle\langle \eta, (\theta(s) - \phi(s))\xi\rangle \\
&= \int_0^T ds\, \mathrm{Re}(\langle ((p_{\alpha\beta}(s, \theta(s)) - q_{\alpha\beta}(s, \phi(s))) \otimes \mathbb{1})(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\theta(s), \\
&\qquad \phi(s))\rangle_{(2)}) \\
&\geq 0 \qquad [\text{since } \mathcal{P} = P \otimes \mathbb{1} \text{ is } monotone]
\end{aligned}$$

Furthermore, $\tilde{P}$ is *maximal monotone*. To see this, suppose that $\theta(\cdot)(\eta, \xi) \in L^2_{\eta\xi}([0, T))$, p: $[0, T) \to \mathrm{sesq}(\mathbb{D} \underline{\otimes} \mathbb{E})$ and

$$\mathrm{Re}(\langle p(\cdot)(\eta, \xi)) - q(\phi(\cdot)(\eta, \xi)), \theta(\cdot)(\eta, \xi) - \phi(\cdot)(\eta, \xi)\rangle_{L^2([0,T))}) \geq 0$$

for all $\phi(\cdot)(\eta, \xi) \in L^2_{\eta\xi}([0, T))$ and $q(\phi(\cdot)(\eta, \xi)) \in \tilde{P}(\phi(\cdot)(\eta, \xi))$. Then,

$$\int_0^T ds\, \mathrm{Re}(\langle ((p(s) - q_{\alpha\beta}(s, \phi(s))) \otimes \mathbb{1})(\xi \otimes \eta), \Phi_{(\eta,\xi)}(\theta(s), \phi(s))\rangle_{(2)}) \geq 0$$

for all $\phi(\cdot)(\eta, \xi) \in L^2_{\eta\xi}([0, T))$ and $q(\phi(\cdot)(\eta, \xi)) \in \tilde{P}(\phi(\cdot)(\eta, \xi))$. Hence, as $\mathcal{P} = P \otimes \mathbb{1}$ is *maximal monotone*, we get $p(s)(\eta, \xi) \in \tilde{P}(\theta(s)(\eta, \xi))$, for almost every $s \in [0, T)$, showing that $\tilde{P}$ is *maximal monotone.*

Next, by Proposition 5.4, $\{P_\lambda(\cdot, \varphi_\lambda(\cdot))(\eta, \xi)\colon \lambda \in (0, 1]\}$ is a bounded subset of $L^2_{\eta\xi}([0, T))$. It follows that a subsequence $\{P_{\lambda_n}(\cdot, \varphi_{\lambda_n}(\cdot))(\eta, \xi)\colon n \in \mathbb{N}\}$ of this net converges weakly in $L^2_{\eta\xi}([0, T))$ to some $\omega(\cdot)(\eta, \xi)$ as $n \to \infty$ with $\lambda_n \downarrow 0$. Also, as we have seen above, the net $\{J_\lambda(\cdot, \varphi_\lambda(\cdot))(\eta, \xi)\colon \lambda \in (0, 1]\}$ converges in $L^2_{\eta\xi}([0, T))$ to $\varphi(\cdot)(\eta, \xi)$ as $\lambda \downarrow 0$. As $\tilde{P}$ is maximal monotone, $\omega(s)(\eta, \xi) \in \tilde{P}(\varphi(s)(\eta, \xi))$, or equivalently, $\omega(s)(\eta, \xi) \in P(s, \varphi(s))(\eta, \xi)$ for almost every $s \in [0, T)$.

Finally, for arbitrary $\vartheta \in L^2_{\eta\xi}([0, T))$, we get

$$\int_0^T dt\, \vartheta(t)(\langle\eta, \varphi_{\lambda_n}(t)\xi\rangle - \langle\eta, x_0\xi\rangle) = \int_0^T dt\, \vartheta(t) \int_0^t ds \frac{d}{ds} \langle\eta, \varphi_{\lambda_n}(s)\xi\rangle$$

$$= -\int_0^T dt\, \vartheta(t) \int_0^t ds\, P_{\lambda_n}(s, \varphi_{\lambda_n}(s))(\eta, \xi)$$

[since φ_{λ_n} is a solution of $(5.1)_{\lambda_n}$]

Hence, as $\lambda_n \downarrow 0$, this gives

$$\int_0^T dt\, \vartheta(t)(\langle\eta, \varphi(t)\xi\rangle - \langle\eta, x_0\xi\rangle) = -\int_0^T dt\, \vartheta(t) \int_0^t ds\, \omega(s)(\eta, \xi)$$

As ϑ is arbitrary in $L^2_{\eta\xi}([0, T))$, it follows that

$$\langle\eta, \varphi(t)\xi\rangle - \langle\eta, x_0\xi\rangle = -\int_0^t ds\, \omega(s)(\eta, \xi)$$

for all $t \in [0, T)$, whence

$$-\frac{d}{dt}\langle\eta, \varphi(t)\xi\rangle = \omega(t)(\eta, \xi) \in P(t, \varphi(t))(\eta, \xi)$$

for all $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ and almost every $t \in \mathbb{R}_+$, since $T > 0$ was arbitrary. This concludes the proof. ■

Remark. As examples, we show that a large class of quantum stochastic differential inclusions which satisfy the assumptions and conclusion of Theorem 5.6 arise as perturbations of certain quantum stochastic differential *equations* by some multivalued stochastic processes.

Let $V: \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$ [resp. $\omega_j: \mathbb{R}_+ \times \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}, j = 1, 2, 3, 4$] be such that the multivalued stochastic process [resp. stochastic process] $t \mapsto V(X(t))$ [resp. $t \mapsto \omega_j(t, X(t)), j = 1, 2, 3, 4$], $t \in \mathbb{R}_+$, is in $L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})_{\mathrm{mvs}}$ [resp. $L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})$] for all $X \in L^2_{\mathrm{loc}}(\tilde{\mathcal{A}})$.

Then, the quantum stochastic differential *inclusion*

$$dX(t) \in -(V(X(t))\, dt + \omega_1(t, X(t))\, d\Lambda_\pi(t) + \omega_2(t, X(t))\, dA_f(t)$$
$$+ \omega_3(t, X(t))\, dA_g^+(t) + \omega_4(t, X(t))\, dt), \qquad \text{almost all} \quad t \in \mathbb{R}_+ \quad (\star)$$
$$X(0) = x_0 \qquad \text{for some} \quad x_0 \in \tilde{\mathcal{A}}$$

is a perturbation by the multivalued stochastic process V of the quantum stochastic differential *equation*

$$dz(t) = -(\omega_1(t, z(t))\, d\Lambda_\pi(t) + \omega_2(t, z(t))\, dA_f(t) + \omega_3(t, z(t))\, dA_g^+(t) + \omega_4(t, z(t))\, dt), \qquad \text{almost all} \quad t \in \mathbb{R}_+$$

$$z(0) = x_0 \in \tilde{\mathcal{A}}$$

For $\alpha, \beta \in L^\infty_{Y,\text{loc}}(\mathbb{R}_+)$, define the single-valued map

$$p_{\alpha\beta}\colon \quad \mathbb{R}_+ \times \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$$

by

$$p_{\alpha\beta}(t, x) = \mu_{\alpha\beta}(t)\omega_1(t, x) + \nu_\beta(t)\omega_2(t, x) + \sigma_\alpha(t)\omega_3(t, x) + \omega_4(t, x)$$

$(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, using previous notation, and the multifunction

$$P_{\alpha\beta}\colon \quad \mathbb{R}_+ \times \tilde{\mathcal{A}} \to 2^{\tilde{\mathcal{A}}}$$

by

$$P_{\alpha\beta}(t, x) = V(x) + p_{\alpha\beta}(t, x), \qquad (t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$$

Problem ($\star$) is equivalent to the nonclassical differential inclusion

$$\frac{d}{dt}\langle \eta, X(t)\xi\rangle \in -P(t, X(t))(\eta, \xi), \qquad \text{almost all} \quad t \in \mathbb{R}_+ \qquad (\star)_P$$

$$X(0) = x_0 \in \tilde{\mathcal{A}}$$

where

$$P(t, x)(\eta, \xi) = \langle \eta, P_{\alpha\beta}(t, x)\xi\rangle, \qquad (t, x)\mathbb{R}_+ \times \tilde{\mathcal{A}}$$

for arbitrary $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$ with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{Y,\text{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$.

There is now the following result.

Theorem 5.7. Suppose that the multifunction P in Problem $(\star)_P$ is such that $\mathcal{P} = P \otimes 1$ is in Hypmax($\mathbb{R}_+ \times \tilde{\mathcal{A}}$), and there are a monotone increasing function $\Psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ and a collection $\{c_{\eta\xi}\colon \eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}\}$ of continuous functions from $\mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|p_{\alpha\beta}(t + s, J_{\lambda,\alpha\beta}(t, x)) - p_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t, x))\|_{\eta,\xi} \leq c_{\eta\xi}(t)s(\|J_{\lambda,\alpha\beta}(t, x)\|_{\eta,\xi} + \Psi(\|x\|_{\eta,\xi}))$$

where $J_{\lambda,\alpha\beta}(t, \cdot)$, $\lambda > 0$, is the resolvent of $P_{\alpha\beta}$, for all $(t, x) \in \mathbb{R}_+ \times \tilde{\mathcal{A}}$, $s \in (0, 1]$, and $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^\infty_{Y,\text{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Then, Problem ($\star$) has a unique adapted solution.

Proof. We only need to check that the inequality of Proposition 5.4 is satisfied under the present assumptions. To this end, let $\eta, \xi \in \mathbb{D} \underline{\otimes} \mathbb{E}$, with $\eta = u \otimes e(\alpha)$, $\xi = v \otimes e(\beta)$, $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$, $u, v \in \mathbb{D}$. Then,

$$\begin{aligned}
&\|J_{\lambda,\alpha\beta}(t+s,x) - J_{\lambda,\alpha\beta}(t,x)\|_{\eta,\xi} \\
&\quad = \|J_{\lambda,\alpha\beta}(t+s,x) - J_{\lambda,\alpha\beta}(t+s, x + \lambda[p_{\alpha\beta}(t+s, J_{\lambda,\alpha\beta}(t,x)) \\
&\qquad - p_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t,x))])\|_{\eta,\xi} \\
&\quad \leq \lambda \|p_{\alpha\beta}(t+s, J_{\lambda,\alpha\beta}(t,x)) - p_{\alpha\beta}(t, J_{\lambda,\alpha\beta}(t,x))\|_{\eta,\xi} \\
&\quad \leq \lambda c_{\eta\xi}(t) s(\|J_{\lambda,\alpha\beta}(t,x)\|_{\eta,\xi} + \Psi(\|x\|_{\eta,\xi}))
\end{aligned}$$

by the assumed inequality and since $x \mapsto J_{\lambda,\alpha\beta}(t,x)$ is Lipschitzian with Lipschitz constant 1, for each $t \in \mathbb{R}_+$, $\lambda > 0$, and arbitrary $\alpha, \beta \in L^{\infty}_{\gamma,\mathrm{loc}}(\mathbb{R}_+)$. As this is the inequality of Proposition 5.4, we are done. ■

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REFERENCES

Aubin, J.-P., and Cellina, A. (1984). *Differential Inclusions*, Springer-Verlag, Berlin.

Browder, F. E. (1976). *Nonlinear Operators and Nonlinear Equations of Evolution in Banach Spaces* (Proceedings of Symposia in Pure Mathematics Volume XVIII, Part 2), American Mathematical Society, Providence, Rhode Island.

Deimling, K. (1992). *Multivalued Differential Equations*, Walter de Gruyter, Berlin.

Ekhaguere, G. O. S. (1992). *International Journal of Theoretical Physics*, **31**, 2003–2027.

Guichardet, A. (1972). *Symmetric Hilbert Spaces and Related Topics*, Springer-Verlag, Berlin.

Hewitt, E., and Stromberg, K. (1965). *Real and Abstract Analysis*, Springer-Verlag, Berlin.

Hudson, R. L., and Parthasarathy, K. R. (1984). *Communications in Mathematical Physics*, **93**, 301–324.

Kisielewicz, M. (1991). *Differential Inclusions and Optimal Control*, Kluwer Academic Publishers, Dordrecht.

Michael, E. (1956). *American Mathematical Monthly*, **63**, 233–238.

Parthasarathy, T. (1972). *Selection Theorems and Their Applications*, Springer-Verlag, Berlin.

Reed, M., and Simon, B. (1972). *Methods of Modern Mathematical Physics: I: Functional Analysis*, Academic Press, New York.

Walter, W. (1964). *Differential- und Integral-Ungleichungen und ihre Anwendung bei Abschätzungs- und Eindeutigkeitsproblemen*, Springer-Verlag, Berlin.